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Flexural Stiffness of Reinforced Concrete Columns and Beams: Analytical Approach

by Madhu Khuntia and S. K. Ghosh

The present ACI code (ACI 318-02) provisions on effective stiffnesses of beams and columns are reviewed. Factors influencing the moments of inertia of beams and columns are discussed. The primary variables considered are: the reinforcement ratio, the axial load ratio, the eccentricity ratio, and the compressive strength of concrete. On the basis of a parametric study, simple formulas are proposed to determine the effective stiffnesses of reinforced concrete columns and beams. The proposed stiffness expressions are applicable for all levels of applied loading, including both service and ultimate loads. The analytical results show that the flexural stiffness assumption in the current ACI code procedure for design of slender columns using the moment magnifier method (Eq. (10-11) and (10-12)) is extremely conservative. Recommendations are made concerning stiffness assumptions in the analysis of reinforced concrete frames under lateral loads.

Keywords: beam; column; flexural stiffness; moment.

INTRODUCTION

Traditionally, design engineers use rough estimates of flexural stiffnesses *EI* of beams and columns in the analysis of reinforced concrete building structures under lateral loads. The use of 1/2 the gross moment of inertia for beams and the full gross moment of inertia for columns is quite common. In view of the availability of second-order analysis as a design tool, advances in the knowledge of structural behavior and loads and initiatives aimed toward the development of multilevel performance-based design methods, it is felt necessary to re-evaluate the traditional stiffness assumptions. It may be noted that Section 10.11.1 of the 2002 edition of the ACI 318 Building Code suggests effective flexural stiffnesses for reinforced concrete structural members, but only for analysis undertaken for the purposes of slender column design. ACI 318 Section 10.11.1 recommends the use of $0.35I_g$ and $0.70I_g$ for beams and columns, respectively, for first-order analysis. Section 10.13.4.1 recommends the use of the same stiffnesses for the second-order analysis of sway frames. No specific recommendations are made concerning effective flexural stiffnesses for general frame analysis. As can be seen, the recommended moment of inertia (*I*)-value for columns is independent of the reinforcement ratio, the axial load, and the eccentricity (bending moment to axial load ratio). Investigations by various researchers, however, show dependence of column flexural stiffness on the level of axial load (Mehanny, Kuramoto, and Deierlein 2001) as well as on the eccentricity ratio (Lloyd and Rangan 1996; Mirza 1990). Similarly, the recommended *I*-value for beams does not take the effect of reinforcement ratio into account. This simplification may not be appropriate in many practical cases.

The assumed stiffnesses of beams and columns can affect structural analysis and design in two significant ways:

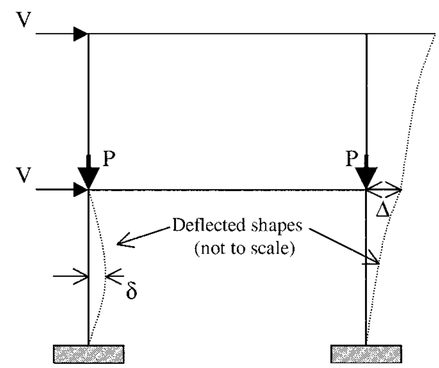

Fig. 1—P-δ and P-Δ effects in buildings.

1. *P-δ and P-Δ effects*: The assumption of lower stiffnesses for columns increases the computed P-δ effect on individual columns and the computed P-Δ effects on an entire story (Fig. 1), thereby substantially enhancing the secondary moments. In many reinforced concrete buildings with a significant number of slender columns, the current ACI procedure using the moment magnifier method (Section 10.13.4.3) predicts a stability failure by P-Δ effects. It is interesting to note that such failure may not be predicted if the other procedures of the ACI code (Section 10.13.4.1 or 10.13.4.2) are employed. Two important points must be noted in the present ACI 318-02 Code. First, different member stiffnesses are recommended to calculate different moment magnifiers. For example, in sway frames, to calculate the stability index Q in Eq. (10-17), it is recommended to use a flexural stiffness of $0.7E_cI_g$ (per Section 10.11.1), whereas to calculate P_{cs} (the critical buckling load) in Eq. (10-18), the recommended *EI* is approximately $0.4E_cI_g$ (Section 10.12.3), which does not seem to be rational. Second, the ACI code uses a factor of 0.75 attached to P_{cs} (in Eq. (10-18)) to account for variability in *EI* and strength. Therefore, the magnification factor obtained by using $\Sigma(P_{us}/0.75P_{cs})$ is overly conservative. For example, in Example 11.2 of PCA Notes to ACI 318-02 (Portland Cement Association 2002), the authors found a moment magnifier by using ($\delta_s = 1/(1 - \Sigma(P_{us}/0.75P_{cs}))$) approximately 20% higher than that by using ($\delta_s = 1/(1 - Q)$). However, by using identical *EI* in calculating both P_{cs} and Q and deleting the factor of 0.75 in the ACI equation, the results from both methods are almost the same, which is expected. In other words, unless formulations are changed, the

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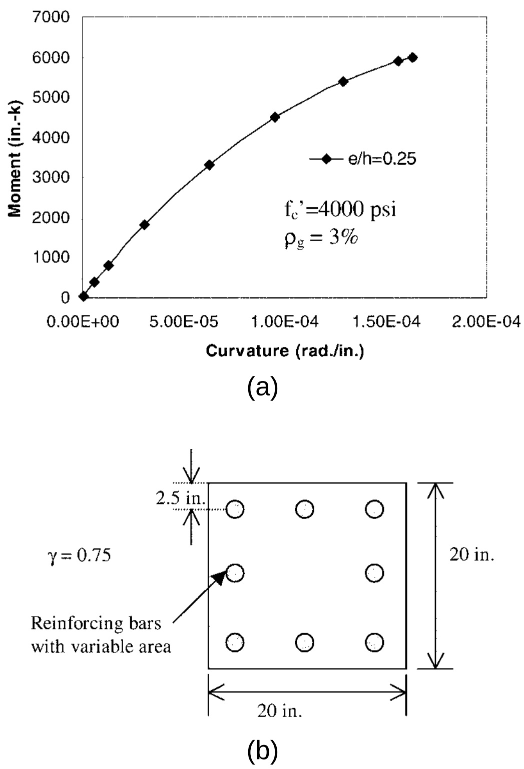

Fig. 2—(a) Moment-curvature relationship for typical reinforced concrete column considered in parametric study (for e/h *ratio of 0.25 and reinforcement ratio of 3%); and (b) cross section of typical reinforced concrete column considered in parametric study.*

moment magnifier calculated using Eq. (10-18) would always be larger than that computed using Eq. (10-17); and

2. *Internal force distribution*—For regular structures, it is generally assumed that the internal force distribution is negligibly affected by the stiffness assumptions of various structural members. This is more or less the case only as long as the column-to-beam stiffness ratios at the various joints remain essentially the same as a structure is subjected to increasing loads. Even in those cases, the P-Δ effects can enhance the internal forces at column ends. Analyses carried out by the authors suggested that the assumption of improper moments of inertia for frame members can lead to unconservative results. For example, it can be shown that the maximum end moments in the columns of a one-story one-bay frame under gravity loads based on an assumption of column $I = 1.0I_g$ and beam $I = 0.3I_g$ (which gives a column-to-beam I/L ratio of 6.6 [with a column height-to-beam span ratio of 2]) can be over 30% larger than those based on an assumption of column $I = 0.5I_g$ and beam $I = 0.5I_g$ (which gives column-to-beam I/L ratio of 2.0). Analyses (as illustrated later in this paper) show that the column I can vary from $0.5I_g$ to $1.0I_g$ and the beam I from $0.3I_g$ to $0.5I_g$ in most practical cases.

The previous discussion leads to the conclusion that more realistic EI values are needed for frame analysis in general and for analysis of frames containing slender columns in particular.

The purpose of this paper is to suggest simplified but reasonably accurate expressions for the computation of effective moments of inertia of beams and columns. The influences of the reinforcement ratio—ρ for beams or ρ_g for columns, the concrete compressive strength f_c', the magnitude of the axial load P, and the eccentricity ratio e/h of the axial load—have been accounted for. This paper shows that the influence of these parameters on the effective moments of inertia of beams and columns can be quite substantial and should not be ignored. Axial load-bending moment histories of slender columns (for a given initial M/P ratio), based on the proposed stiffness assumptions, are compared with many test results in a companion paper (Khuntia and Ghosh 2004) and are found to be in good agreement.

This paper has three parts: 1) an expression for the moments of inertia of reinforced concrete columns is derived using a parametric study, and the influences of various important parameters are pointed out; 2) an expression for the moments of inertia of reinforced concrete beams is derived using a parametric study and compared with traditional analytical results based on the transformed area concept; and 3) a brief review of current ACI code provisions concerning EI in general and EI for slender columns in particular is provided. The need for modifications to the current code provisions is explained.

RESEARCH SIGNIFICANCE

This paper is related to the work of the Slender Column Task Group of ACI Committee 318, Structural Concrete Building Code. The task group is trying to formulate code provisions to streamline and, if possible, simplify the requirements of ACI 318, Sections 10.11 to 10.13, on slender column design. One of the major elements in slender column design is a suitable assumption of flexural stiffness EI of the column.

FLEXURAL STIFFNESS OF REINFORCED CONCRETE COLUMNS

Parametric study

A parametric study was undertaken to investigate the influence of various parameters on the effective EI of reinforced concrete columns. The primary variables were: reinforcement ratio ρ_g (1 and 3%); the concrete compressive strength f_c' (4000 and 12,000 psi); the axial load ratio P/P_o (ranging from 0.00 to 0.80); and the eccentricity ratio, e/h or M/Ph (ranging from 0.10 to 0.80). It may be noted that these ranges encompass almost all practical columns. For example, the magnitude of P_u/P_o is not allowed by the ACI code to exceed 0.64 for any column ($0.64 = 0.75 \times 0.85$ using a value of ϕ = 0.75 for spirally reinforced columns from Section C.3.2.2(a) in Eq. (10-1) of ACI 318-02). Similarly, when e/h exceeds 0.8 to 1.0, the magnitude of $P_u/A_g f_c'$ is not expected to exceed 0.10, thereby allowing the member to be treated as a beam.

In the parametric study, the effective moment of inertia is calculated as the ratio of bending moment over curvature ($EI_e = M/\phi$), as illustrated in Fig. 2(a). It should be emphasized that the magnitude of EI_e is computed up to the yielding of tensile reinforcement, as the value would drastically diminish after steel yielding and is of little importance for frame analysis. In addition, reinforcement at column ends is unlikely to yield in a structure designed using the strong column-weak beam concept. More importantly, the parametric study, as shown

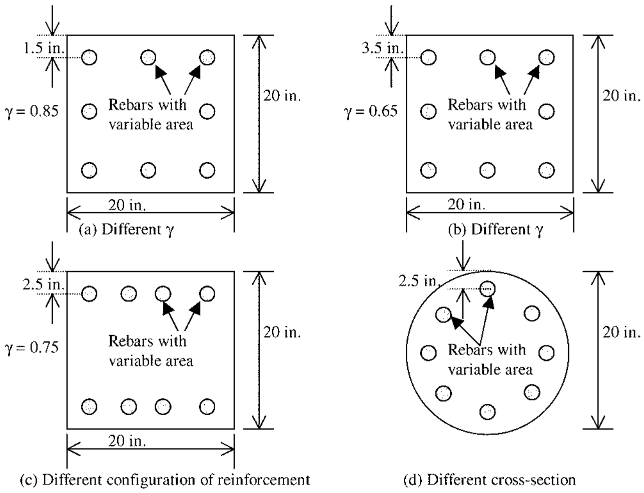

Fig. 3—Cross section of typical reinforced concrete columns considered in parametric study.

later, shows that the column reinforcement in tension is unlikely to yield under any design loading conditions.

A square column of 20 x 20 in. plan dimensions, with 2.5 in. cover to the center of longitudinal reinforcement, is considered for the parametric study (Fig. 2(b)). For this section, the value of γ chosen = 15/20 = 0.75, where γ is the ratio of center-to-center distance between the outermost bars to the overall dimension of the section. However, other reinforcement configurations, γ-values and cross sections, as shown in Fig. 3, are also considered to investigate their effects. The yield strength of reinforcing steel is assumed to be 60 ksi. The axial load-moment interaction diagrams (Fig. 4) are plotted using the ACI rectangular stress block (uniform stress over stress block = $0.85f_c'$) for nominal and design strengths. However, a more exact parabolic stress-strain curve based on Eq. (1) (Fig. 5) is used for plotting the radial lines.

$$\frac{f_c}{f_c'} = 2\left(\frac{\varepsilon_c}{\varepsilon_o}\right) - \left(\frac{\varepsilon_c}{\varepsilon_o}\right)^2 \tag{1}$$

where f_c = compressive stress at a concrete strain of ε_c. The ultimate failure strain of concrete in compression (ε_u) is assumed to be 0.003. ε_o, the strain corresponding to $f_c = f_c'$, is taken to be 0.002 for normal-strength concrete (f_c' = 4000 psi in the study) and 0.0024 for high-strength concrete (f_c' = 12,000 psi in the study). Analyses by the authors using slightly different ε_o and ε_u for high-strength concrete produced very insignificant changes to the analytical results. As will be shown later, the predictions of flexural stiffness for columns with high-strength concrete is generally on the conservative side compared with those for columns with normal-strength concrete. For simplicity, Eq. (1) is used for both normal- and high-strength concrete.

Note that the nominal strength interaction diagram is drawn for a concrete compressive strain of 0.003 only, whereas the stress-strain curve given by Eq. (1) can be used to illustrate complete loading histories.

As can be seen from Fig. 5, the elastic tangent modulus of concrete diminishes significantly for compressive strains beyond 0.0015 (approximately). In other words, when a portion of column is strained beyond a compressive strain of 0.0015, the effective elastic modulus of that portion is comparatively lower than those of portions having lesser compressive strains.

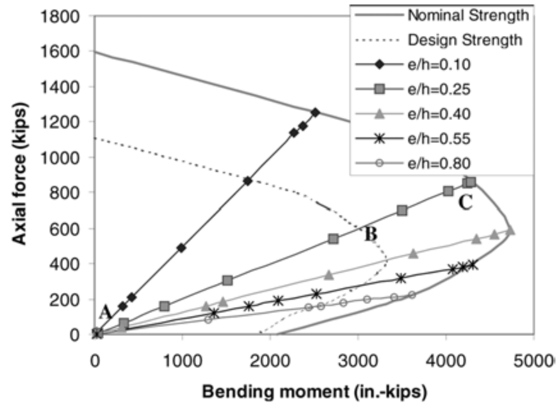

Fig. 4—P-M interaction diagram and loading histories for typical column (ρ_g = 1% and f_c' = 4000 psi).

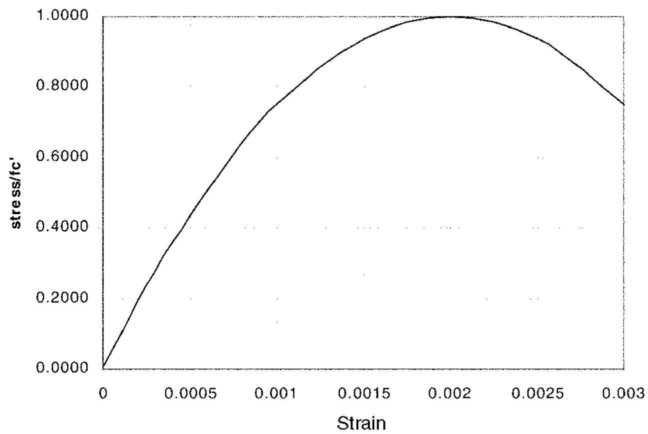

Fig. 5—Typical stress-strain curve for concrete in compression.

Figure 4 shows column strength interaction curves for a 1% gross reinforcement ratio and a concrete strength of 4000 psi. The radial lines show *P-M* combinations for various eccentricity ratios e/h for short columns. As can be seen from Fig. 4, the member may be treated as a beam when the e/h ratio exceeds approximately 0.8. For e/h > 0.8, the design axial strength ϕP_n is most likely less than $0.1A_g f_c'$ (= 160 kips for the case considered) and the behavior is mainly flexural.

Figure 6 shows the variation of EI_e/E_cI_g for the column with changes in the eccentricity ratio e/h and the axial load ratio P_u/P_o. I_g is the gross moment of inertia of section (equal to $bh^3/12$ for a rectangular section with width b and total depth h). The elastic modulus of concrete E_c is given in ACI 318 (Section 8.5.1) as

$$E_c = 33w_c^{1.5}\sqrt{f_c'} \tag{2}$$

In Eq. (2), both E_c and f_c' are in psi. It may be noted that the use of other appropriate expressions for high-strength concrete does not change the basic concept outlined in this paper. As will be shown later, the moment of inertia of columns using high-strength concrete is generally higher than that of columns using normal-strength concrete. Therefore, the use of somewhat lower elastic modulus for high-strength concrete does not affect the magnitude of effective flexural stiffness EI_e significantly. The primary aim of the paper is to predict the effective flexural stiffness EI_e in terms of E_cI_g of columns. Therefore,

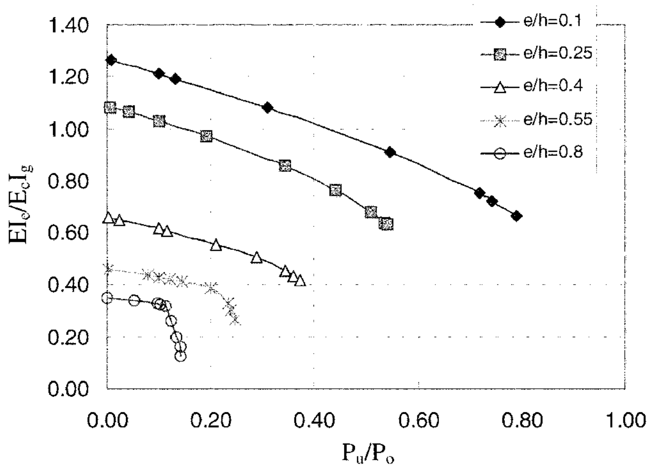

Fig. 6—Influence of eccentricity ratio and axial load ratio on effective I *of columns (ρ_g = 1% and f_c' = 4000 psi).*

Eq. (2), which is given in ACI Section 8.5.1, is used for both high- and normal-strength concrete for simplicity, without any appreciable loss of accuracy.

Table 1 shows the influence of the reinforcement ratio ρ_g, the axial load ratio P_u/P_o and the eccentricity ratio e/h on the effective moment of inertia of the column of Fig. 2(b), based on a parametric study using normal-strength concrete (f_c' = 4000 psi). Table 2 shows the influences for high-strength concrete (f_c' = 12,000 psi).

From the analyses, it is found that the maximum compressive strain in concrete that can occur corresponding to the design strength curve (which is plotted by reducing the nominal strength curve by ϕ = 0.7 as per Section C.3.2.2 of the 2002 ACI Code for columns with factored axial load more than $0.1A_g f_c'$) does not exceed 0.0015. Table 1 (Column 7) shows the compressive strain in concrete for different e/h and P/P_o (or P_u/P_o) ratios prior to yielding of tension or compression steel. Drastic reductions in EI_e occur between the design and the nominal strength curves. This is because of the existence of higher compressive strains in concrete (more than 0.0015, for example), which leads to lower tangent elastic modulus of the concrete (refer to Fig. 5). In other words, for any e/h ratio, the effective EI of a column is significantly larger from Point A to B (that is, within design strength curve) than from Point B to C (that is, from design to nominal strength curve) (refer to Fig. 4). It may be emphasized that for reinforced concrete (RC) columns, where applied forces (M_u, P_u) are within the design strength curve, the maximum strain on the compression side does not exceed 0.0015. Thus, there is little chance of yielding of the compression steel (yield strain of steel is 0.0021 for 60 ksi reinforcing bars). In addition, the maximum tensile strain in the reinforcement also seldom, if ever, exceeds its yield strain. Note that: 1) above the balanced point, tensile steel does not yield before the concrete crushes in compression; and 2) below the balanced point and with P_u more than $0.1A_g$, the parametric study shows that the tensile steel is unlikely to yield for applied loads within the design strength curve. Therefore, the effective EI of code-conforming columns is not expected to decrease significantly, which can happen in the case of reinforced concrete beams. It is worthwhile to point out that for frame analysis it is certainly conservative to use effective EI corresponding to the design strength curve, not corresponding to the nominal strength curve, because the required strength from all the load combinations must lie within the design strength curve.

Analysis shows that the moment of inertia of a column depends on four major factors, as discussed below.

Influence of reinforcement ratio ρ_g—When the gross steel area increases, for a particular neutral axis depth, the axial load and the corresponding bending moment increase for any particular P_u/P_o and e/h ratios. Therefore, the effective EI (obtained by dividing M with ϕ) is substantially higher for columns with higher steel ratios. Table 1 and 2 show that the EI_e of a column is always higher for higher reinforcement ratios for any particular axial load ratio and eccentricity ratio. For example, Table 1 shows that the EI_e of a column with a 3% steel ratio is approximately 28% higher than that of a column with a 1% steel ratio at an e/h ratio of 0.25 and for P_u/P_o equal to 0.44 (see the bold rows in Table 1).

Influence of eccentricity ratio e/h—This is the second most important factor affecting the EI of a column. For columns with high e/h (or M/Ph) ratios, the bending moment is higher for a given axial load, leading to an increase in flexural crack length and reduction in effective EI of the section. The reduction of EI_e due to an increase in e/h is also reported by Mirza (1990) and Lloyd and Rangan (1996). Table 1 and 2 show the influence of increasing e/h on the magnitude of EI. Graphically, Fig. 6 shows that for any axial load ratio (P_u/P_o), EI_e decreases with increasing e/h. It should be emphasized that increasing e/h beyond 0.8 allows the member to be treated as a beam, as the factored axial load never exceeds $0.1A_g f_c'$.

Influence of axial load ratio P_u/P_o—When the axial load P_u increases, the depth of flexural cracks decreases. Therefore, it is to be expected that the effective EI of a column should increase with P_u/P_o. Analysis shows, however, that for a given e/h ratio, when P_u (and the corresponding M_u) increases, the compressive strain in the concrete at extreme fiber increases in higher proportion than an increase in P_u/P_o. For example, when P_u (and the corresponding M_u) increases by 50%, the corresponding increase in ε_c (and the corresponding ϕ) is much more than 50% (80%, for example). Therefore, an increase in P_u/P_o ratio always results in a reduction in effective EI for a column. Note that: a) only the effect of P_u/P_o is considered herein; and b) the value of P_o is different for two similarly sized columns with different reinforcement ratios.

Table 1 and 2 show the influence of increasing P_u/P_o on the magnitude of EI_e for different e/h ratios. Graphically, the radial lines in Fig. 4 show the increase in P_u/P_o at various e/h ratios. Figure 6 shows that for any eccentricity ratio e/h, EI_e decreases with increasing P_u/P_o.

To summarize the effect of the P_u/P_o ratio and the e/h ratio, Fig. 7 is drawn showing a column strength interaction diagram. Line A-B shows a gradual increase of e/h ratios at a constant axial load ratio. As explained previously, the effective EI would decrease with increasing e/h ratios, that is, from A to B. Line C-B shows a gradual increase of P_u/P_o ratios at a constant eccentricity ratio e/h. As explained previously, the effective EI would decrease with increasing P_u/P_o ratios, that is, from C to B.

Influence of high-strength concrete—The parametric study shows that an increase in concrete strength increases the effective EI of a column for given P_u/P_o and e/h ratios (Table 1 and 2). For example, the results show that the EI_e of a column having a 1% gross reinforcement ratio and a compressive concrete strength of 12,000 psi at a P_u/P_o ratio

Table 1—Influence of various parameters on effective *EI* of reinforced concrete columns (f_c' = 4000 psi)

	Analysis			Proposed EI_e/E_cI_g						
ρ_g, % [1]	EI_e/E_cI_g [2]	P_u/P_o [3]	e/h [4]	Eq. (3) [5]	Ratio [6] = [2]/[5]	ε_c [7]	Eq. (5) [8]	Ratio [9] = [2]/[8]	Eq. (6) [10]	Ratio [11] = [2]/[10]
1	1.212	0.100	0.10	0.89	1.36	0.00015	0.63	**1.92**	0.37	3.29
1	1.193	0.132	0.10	0.88	1.36	0.0002	0.63	**1.89**	0.38	3.10
1	1.082	0.310	0.10	0.78	1.38	0.0005	0.63	**1.72**	0.48	2.26
1	0.908	0.547	0.10	0.66	1.38	0.001	0.63	**1.44**	0.60	1.51
1	0.750	0.719	0.10	0.57	1.32	0.0015	0.63	**1.19**	0.69	1.08
1	0.722	0.744	0.10	0.55	1.30	0.0016	0.63	**1.15**	0.71	1.02
1	0.665	0.789	0.10	0.53	1.25	0.0018	0.63	**1.06**	0.73	0.91
1	1.030	0.101	0.25	0.74	1.40	0.00025	0.55	**1.87**	0.37	2.80
1	0.970	0.191	0.25	0.69	1.41	0.0005	0.55	**1.76**	0.42	2.34
1	0.855	0.343	0.25	0.61	1.41	0.001	0.55	**1.55**	0.49	1.73
1	**0.763**	**0.442**	**0.25**	**0.56**	**1.37**	0.00142	**0.55**	1.38	**0.55**	**1.40**
1	0.682	0.508	0.25	0.52	1.31	0.0018	0.55	**1.24**	0.58	1.17
1	0.640	0.535	0.25	0.51	1.26	0.002	0.55	**1.16**	0.60	1.07
1	0.630	0.541	0.25	0.50	1.25	0.00205	0.55	1.14	0.60	1.05
1	0.615	0.101	0.40	0.58	1.06	0.00043	0.47	**1.30**	0.37	1.67
1	0.607	0.116	0.40	0.57	1.06	0.0005	0.47	**1.28**	0.38	1.62
1	0.555	0.211	0.40	0.52	1.07	0.001	0.47	**1.17**	0.43	1.31
1	0.509	0.289	0.40	0.48	1.06	0.0015	0.47	**1.07**	0.47	1.09
1	0.456	0.343	0.40	0.45	1.01	0.002	0.47	**0.96**	0.49	0.92
1	0.435	0.358	0.40	0.44	0.99	0.0022	0.47	**0.92**	0.50	0.86
1	0.416	0.374	0.40	0.44	0.96	0.0024	0.47	**0.88**	0.51	0.81
1	0.429	0.101	0.55	0.42	1.02	0.00066	0.39	**1.09**	0.37	1.17
1	0.422	0.120	0.55	0.41	1.03	0.0008	0.39	**1.07**	0.38	1.12
1	0.413	0.145	0.55	0.40	1.04	0.001	0.39	**1.05**	0.39	1.06
1	0.388	0.200	0.55	0.37	1.05	0.0015	0.39	**0.98**	0.42	0.92
1	0.325	0.101	0.80	0.23	1.44	0.00107	0.26	**1.24**	0.37	0.88
1	0.321	0.112	0.80	0.23	1.43	0.0012	0.26	**1.22**	0.37	0.86
3	1.426	0.078	0.10	1.00	1.43	0.00013	0.93	**1.53**	0.53	2.71
3	1.398	0.119	0.10	1.00	1.40	0.0002	0.93	**1.50**	0.56	2.51
3	1.286	0.279	0.10	1.00	1.29	0.0005	0.93	**1.38**	0.68	1.89
3	1.107	0.504	0.10	1.00	1.11	0.001	0.93	**1.19**	0.86	1.29
3	0.943	0.674	0.10	0.87	1.08	0.0015	0.93	**1.01**	0.99	0.95
3	0.793	0.794	0.10	0.78	1.02	0.002	0.93	**0.85**	1.00	0.79
3	1.276	0.078	0.25	1.00	1.28	0.00021	0.81	**1.57**	0.53	2.43
3	1.208	0.177	0.25	1.00	1.21	0.0005	0.81	**1.48**	0.60	2.01
3	1.092	0.324	0.25	0.91	1.20	0.001	0.81	**1.34**	0.72	1.53
3	**0.979**	**0.441**	**0.25**	**0.82**	**1.19**	0.0015	**0.81**	1.20	**0.81**	**1.21**
3	0.870	0.529	0.25	0.75	1.16	0.002	0.81	**1.07**	0.87	1.00
3	0.785	0.58	0.250	0.71	1.10	0.0024	0.81	**0.97**	0.91	0.86
3	0.801	0.079	0.55	0.63	1.26	0.00045	0.58	**1.38**	0.53	1.52
3	0.781	0.118	0.55	0.60	1.29	0.0007	0.58	**1.35**	0.56	1.40
3	0.757	0.163	0.55	0.57	1.32	0.001	0.58	1.30	0.59	1.28
3	0.713	0.226	0.55	0.52	1.37	0.0015	0.58	**1.23**	0.64	1.11
3	0.667	0.278	0.55	0.48	1.39	0.002	0.58	**1.15**	0.68	0.98
3	0.620	0.317	0.55	0.48	1.30	0.0025	0.58	**1.07**	0.71	0.87
3	0.687	0.078	0.80	0.48	1.45	0.00066	0.48	**1.45**	0.53	1.31
3	0.667	0.114	0.80	0.48	1.40	0.001	0.48	**1.40**	0.55	1.21
3	0.656	0.134	0.80	0.48	1.38	0.0012	0.48	**1.38**	0.57	1.15
3	0.637	0.160	0.80	0.48	1.34	0.0015	0.48	**1.34**	0.59	1.08
3	0.605	0.199	0.80	0.48	1.27	0.002	0.48	**1.27**	0.62	0.98
					Mean = 1.24			Mean = 1.28		Mean = 1.42
					s.d. = 0.5			s.d. = 0.26		s.d. = 0.63

Note: Ratio means ratio of parametric to proposed equation.

of 0.44 and an e/h ratio of 0.25 is approximately 32% more than that of the same column with a compressive strength of 4000 psi (refer to bold rows of Table 1 and 2). The reason is as follows: for a given P_u/P_o ratio, P_u is substantially higher for high-strength concrete (as P_o is high). This provides a higher M_u for a given e/h ($M_u = P_ue$). In addition, the neutral axis depth c_u does not change appreciably with an increase in concrete strength, as happens in the case of a beam. Therefore, compared with a lower-concrete-strength column, a high-strength column can carry more bending moment (M or M_u) at a similar curvature ($\phi = \varepsilon_c/c$), leading to a higher effective *EI* for the latter ($EI_e = M/\phi$). Analyses show that the

Table 2—Effective *EI* of high-strength reinforced concrete columns (f_c' = 12,000 psi)

	Analysis			Proposed EI_e/E_cI_g					
ρ_g, % [1]	EI_e/E_cI_g [2]	P_u/P_o [3]	e/h [4]	Eq. (3) [5]	Ratio [6] = [2]/[5]	Eq. (5) [7]	Ratio [8] = [2]/[7]	Eq. (6) [9]	Ratio [10] = [2]/[9]
1	1.655	0.112	0.10	0.89	1.87	0.63	2.63	0.37	4.43
1	1.516	0.278	0.10	0.80	1.90	0.63	2.41	0.46	3.29
1	1.305	0.501	0.10	0.68	1.91	0.63	2.07	0.58	2.26
1	1.110	0.671	0.10	0.59	1.87	0.63	1.76	0.67	1.66
1	0.934	0.790	0.10	0.53	1.76	0.63	1.48	0.73	1.28
1	1.371	0.112	0.25	0.73	1.88	0.55	2.49	0.37	3.67
1	1.317	0.169	0.25	0.70	1.88	0.55	2.39	0.40	3.26
1	1.176	0.308	0.25	0.63	1.88	0.55	2.13	0.48	2.47
1	**1.008**	**0.442**	**0.25**	**0.56**	**1.81**	**0.55**	**1.83**	**0.55**	**1.84**
1	0.910	0.498	0.25	0.53	1.73	0.55	1.65	0.58	1.58
1	0.809	0.542	0.25	0.50	1.61	0.55	1.47	0.60	1.35
1	0.629	0.112	0.40	0.57	1.11	0.47	1.33	0.37	1.68
1	0.576	0.195	0.40	0.53	1.10	0.47	1.22	0.42	1.38
1	0.545	0.231	0.40	0.51	1.08	0.47	1.16	0.44	1.25
1	0.511	0.274	0.40	0.48	1.05	0.47	1.08	0.46	1.11
1	0.359	0.112	0.55	0.41	0.87	0.39	0.91	0.37	0.96
3	1.776	0.103	0.10	1.00	1.78	0.93	1.91	0.54	3.26
3	1.633	0.268	0.10	1.00	1.63	0.93	1.76	0.67	2.43
3	1.424	0.485	0.10	1.00	1.42	0.93	1.54	0.84	1.69
3	1.230	0.654	0.11	0.88	1.40	0.93	1.33	0.97	1.27
3	1.049	0.781	0.10	0.78	1.34	0.93	1.13	1.00	1.05
3	1.512	0.103	0.25	1.00	1.51	0.81	1.86	0.54	2.78
3	1.455	0.166	0.25	1.00	1.45	0.81	1.79	0.59	2.45
3	1.312	0.304	0.25	0.92	1.43	0.81	1.62	0.70	1.87
3	1.180	0.416	0.25	0.83	1.42	0.81	1.46	0.79	1.50
3	1.058	0.507	0.25	0.77	1.38	0.81	1.30	0.86	1.23
3	0.953	0.554	0.25	0.73	1.31	0.81	1.17	0.89	1.07
3	0.943	0.103	0.40	0.85	1.11	0.70	1.35	0.54	1.73
3	0.915	0.139	0.40	0.82	1.12	0.70	1.31	0.57	1.60
3	0.876	0.190	0.40	0.78	1.13	0.69	1.26	0.61	1.43
3	0.816	0.264	0.40	0.72	1.13	0.70	1.17	0.67	1.22
3	0.755	0.324	0.40	0.67	1.12	0.70	1.09	0.72	1.05
3	0.702	0.359	0.41	0.64	1.09	0.69	1.01	0.74	0.94
3	0.679	0.376	0.40	0.63	1.07	0.69	0.98	0.76	0.90
3	0.686	0.103	0.55	0.62	1.11	0.58	1.18	0.54	1.26
3	0.672	0.133	0.55	0.59	1.13	0.58	1.16	0.57	1.18
3	0.636	0.186	0.55	0.55	1.16	0.58	1.10	0.61	1.04
3	0.618	0.215	0.55	0.53	1.17	0.58	1.06	0.63	0.98
3	0.532	0.199	0.80	0.48	1.12	0.48	1.12	0.54	0.98
					Mean = 1.41		Mean = 1.50		Mean = 1.75
					s.d. = 0.32		s.d. = 0.46		s.d. = 0.87

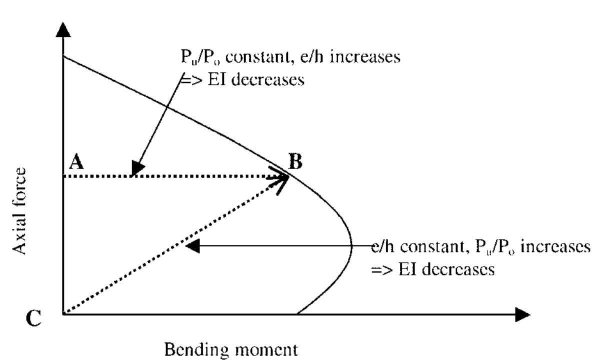

Fig. 7—Influence of axial load and eccentricity ratios on moment of inertia of columns.

computed effective *EI* (that is, EI_e) is approximately proportional to $f_c'^{\,0.67}$, whereas E_cI_g is proportional to $f_c'^{\,0.5}$ (by using ACI 318-02 Eq. (8.5.1) for E_c and I_g being a geometric property of the section). Therefore, the ratio of EI_e/E_cI_g for high-strength columns is higher than that for normal-strength columns.

Proposed expressions for moment of inertia of reinforced concrete columns

Based on the previous parametric study, Eq. (3)—which incorporates the influences of ρ_g, e/h, and P_u/P_o—is proposed to calculate the effective flexural stiffness of a column. For simplicity, the effect of higher concrete strength is conservatively neglected.

$$EI_e = E_cI_g(0.80 + 25\rho_g) \times \left(1 - \frac{e}{h} - 0.5\frac{P_u}{P_o}\right) \quad (3)$$

$$\leq E_cI_g \geq E_cI_{beam}$$

In Eq. (3), the gross reinforcement ratio ρ_g is a decimal fraction.

By the time the designer needs to compute EI_e, the first trial values of most of the terms in Eq. (3) are known. The sizes of the columns and beams have been assumed to carry out the frame analysis. The frame analysis gives values of M_u and P_u, which can be used to estimate e/h and P_u/P_o. Assuming a value of ρ_g, a first trial value of EI_e can be computed.

The lower limit for the effective EI of a column is taken to be the EI_e of an equivalent beam, that is, EI_e of the member when it can be treated as a beam rather than a column. This happens when a member is subjected to a very low axial load and a high e/h ratio ($e/h > 0.8$, for example). For calculating EI_e of an equivalent beam, ρ, the tensile steel ratio (not the gross steel ratio) must be used, which can be approximately taken as half of ρ_g for a column with symmetrical reinforcement. The computation of EI_e of a beam, which varies mainly with the tensile steel ratio, is discussed later. The upper limit is E_cI_g for purposes of conservatism; it can be higher for heavily reinforced columns with low e/h ratios.

Column 6 in Table 1 compares the proposed expression (Eq. (3)) (Column 5) with the results from the parametric study (Column 2). It shows that the prediction by Eq. (3) is quite reasonable and generally on the conservative side. The mean analytical/predicted EI_e/E_cI_g ratio was found to be 1.24 with a standard deviation of 0.15. Only two out of 50 values of the ratio of analytical EI_e to EI_e from Eq. (3) are marginally less than 1.0. It is interesting to note that these two points fall close to the nominal strength curve in Fig. 4. The most conservative predictions correspond to low levels of axial load at a particular eccentricity ratio (that is, near Point A of Fig. 4). Similar comparisons of Eq. (3) with the results from the parametric study of high-strength concrete are presented in Table 2. Equation (3) indirectly allows for high-strength concrete by using E_c from Eq. (2). Table 2 shows the more conservative nature of Eq. (3) for high-strength concrete columns.

The parametric study and analyses by the authors show that the following approximate relationship (Eq. (4)) between P_u/P_o and e/h is quite reasonable when the applied load combinations (P_u, M_u) are close to the design strength curve (Fig. 4)

$$P_u/P_o + e/h = 0.7 \quad (4)$$

Equation (4) can also be verified from Fig. 6 by using $P_u = 0.7P_n$ for any particular e/h ratio. For example, Fig. 6 shows that at $e/h = 0.40$, $P_n/P_o = 0.36$ ($P_u = P_n$ at nominal strength curve), which gives $P_u/P_o = 0.7 \times 0.36 = 0.25$, thereby making the left-hand side of Eq. (4) equal to 0.65. A value of 0.7 is used in Eq. (4) for simplicity.

Using Eq. (4), Eq. (3) can be reduced to either of the following two expressions

$$\text{In terms of } e/h,\ EI_e = E_cI_g(0.80 + 0.25\rho_g) \times \quad (5)$$

$$\left(0.65 - 0.5\frac{e}{h}\right) \leq E_cI_g \geq E_cI_{beam}$$

$$\text{In terms of } P_u/P_o,\ EI_e = E_cI_g(0.80 + 0.25\rho_g) \times \quad (6)$$

$$\left(0.30 - 0.5\frac{P_u}{P_o}\right) \leq E_cI_g \geq E_cI_{beam}$$

The results using Eq. (5) and (6) are compared with those from the parametric study in Table 1 and 2. As can be seen, the predictions by Eq. (5) (in terms of e/h ratio) are slightly more conservative, when compared with the predictions by Eq. (3). The predictions by Eq. (6) (in terms of P_u/P_o) are the least accurate among the three equations (Eq. (3), (5), and (6)).

Note that, strictly speaking, Eq. (5) and (6) are quite accurate if the applied load combinations (P_u, M_u) are close to the design strength curve, which occurs in the worst possible cases. For load combinations located away from the design strength curve, the proposed expressions generally give conservative results. For better accuracy in general, and under service load conditions in particular, it is recommended to use Eq. (3).

Influence of minor parameters

The influences of concrete cover (in terms of γ), reinforcement distribution, and cross-sectional shape were also investigated.

Influence of concrete cover—Figure 3(a) and (b) show two different values of concrete cover to reinforcement, in addition to the base value of 2.5 in. in Fig. 2(b). With a cover of 1.5 in. to the center of longitudinal reinforcement, Fig. 3(a) gives a γ-value of 0.85 and with a cover of 3.5 in. to the center of longitudinal reinforcement, Fig. 3(b) gives a γ-value of 0.65, compared with a γ-value of 0.75 in Fig. 2(b). Table 3 shows the results from the parametric study as well as those using the proposed Eq. (3). It shows that by decreasing γ from 0.75 (Fig. 2(b)) to 0.65 (Fig. 3(b)), the average effective stiffness decreases by approximately 7 to 12%. A similar increase is also observed when γ increases from 0.75 to 0.85 (Fig. 3(a)). In all cases, however, the predicted results are on the conservative side. For simplicity, the influence of γ on effective flexural stiffness can be ignored.

Influence of reinforcement distribution—Figure 3(c) shows a reinforcement configuration different from the base reinforcement configuration (Fig. 2(b)). Note that, in Fig. 2(b), the reinforcement is distributed along the depth, whereas the reinforcement is concentrated near two faces in Fig. 3(c). Table 4 shows the results from the parametric study as well as those using the proposed Eq. (3). It shows that by concentrating the reinforcement near two faces (Fig. 3(c)), the average effective stiffness increases by approximately 8% compared with the case with distributed reinforcement (Fig. 2(b)) for a gross reinforcement ratio of 1%, and by approximately 15% for a gross reinforcement ratio of 3%. In all cases, however, the predicted results are conservative. For simplicity, the effect of reinforcement distribution on effective flexural stiffness can be neglected.

Influence of shape of cross section—Figure 3(d) shows a circular cross section, obviously different from the square cross section of Fig. 2(b). The reinforcement is distributed along the perimeter, as is generally the case in practice. Table 5 shows the results from the parametric study as well as those using the proposed Eq. (3). It shows that the average effective stiffness of a circular section is over 10% less than that of a square section with plan dimension equal to diameter, the same concrete strength, and the same gross reinforcement ratio. In all cases, however, the predicted results are

Table 3—Influence of concrete cover on effective *EI* of columns (f_c' = 4000 psi)

	Analysis			Proposed Eq. (3)		
ρ_g, % [1]	EI_e/E_cI_g [2]	P_u/P_o [3]	e/h [4]	EI_e/E_cI_g [5]	Ratio [6] = [2]/[5]	Various cases
1	1.030	0.101	0.25	0.74	1.40	Case 1 dc = 2.5 in. γ = 0.75 ρ_g = 1%
1	0.970	0.191	0.25	0.69	1.41	
1	0.855	0.343	0.25	0.61	1.41	
1	0.763	0.442	0.25	0.56	1.37	
1	0.682	0.508	0.25	0.52	1.31	
1	0.640	0.535	0.25	0.51	1.26	
1	0.630	0.541	0.25	0.50	1.25	
					Mean = 1.35	
1	1.077	0.101	0.25	0.73	1.47	Case 2 dc = 1.5 in. γ = 0.85 ρ_g = 1%
1	1.018	0.196	0.25	0.69	1.49	
1	0.902	0.352	0.25	0.60	1.50	
1	0.810	0.449	0.25	0.55	1.47	
1	0.724	0.524	0.25	0.51	1.41	
1	0.680	0.552	0.25	0.50	1.37	
1	0.639	0.575	0.25	0.49	1.32	
					Mean = 1.43	
1	0.983	0.102	0.25	0.73	1.34	Case 3 dc = 3.5 in. γ = 0.65 ρ_g = 1%
1	0.924	0.186	0.25	0.69	1.34	
1	0.814	0.334	0.25	0.61	1.33	
1	0.723	0.430	0.25	0.56	1.29	
1	0.641	0.492	0.25	0.53	1.21	
1	0.600	0.517	0.25	0.52	1.16	
1	0.560	0.536	0.25	0.51	1.11	
					Mean = 1.25	
3	1.276	0.078	0.25	1.00	1.28	Case 4 dc = 2.5 in. γ = 0.75 ρ_g = 3%
3	1.208	0.177	0.25	1.00	1.21	
3	1.092	0.324	0.25	0.91	1.20	
3	0.979	0.441	0.25	0.82	1.19	
3	0.870	0.529	0.25	0.75	1.16	
3	0.785	0.579	0.25	0.71	1.10	
					Mean = 1.19	
3	1.395	0.078	0.25	1.00	1.40	Case 5 dc = 1.5 in. γ = 0.85 ρ_g = 3%
3	1.322	0.185	0.25	1.00	1.32	
3	1.203	0.340	0.25	0.90	1.34	
3	1.088	0.466	0.25	0.80	1.36	
3	0.973	0.560	0.25	0.73	1.34	
3	0.928	0.590	0.25	0.70	1.32	
					Mean = 1.35	
3	1.161	0.078	0.25	1.00	1.16	Case 6 dc = 3.5 in. γ = 0.65 ρ_g = 3%
3	1.097	0.168	0.25	1.00	1.10	
3	0.985	0.306	0.25	0.92	1.07	
3	0.878	0.415	0.25	0.84	1.04	
3	0.773	0.496	0.25	0.78	0.99	
3	0.693	0.539	0.25	0.74	0.93	
					Mean = 1.05	

Note: dc = cover to center of longitudinal reinforcement. Refer to Fig. 2(b) and 3(a) and (b). Mean ratio indicates ratio of analytical-to-proposed.

Table 4—Influence of reinforcement distribution on effective *EI* of reinforced concrete columns (f_c' = 4000 psi)

	Analysis			Proposed Eq. (3)		
ρ_g, % [1]	EI_e/E_cI_g [2]	P_u/P_o [3]	e/h [4]	EI_e/E_cI_g [5]	Ratio [6] = [2]/[5]	Various cases
1	1.030	0.101	0.25	0.74	1.40	Case 1 dc = 2.5 in. γ = 0.75 ρ_g = 1% Distributed
1	0.970	0.191	0.25	0.69	1.41	
1	0.855	0.343	0.25	0.61	1.41	
1	0.763	0.442	0.25	0.56	1.37	
1	0.682	0.508	0.25	0.52	1.31	
1	0.640	0.535	0.25	0.51	1.26	
1	0.630	0.541	0.25	0.50	1.25	
					Mean = 1.35	
1	1.086	0.101	0.250	0.73	1.48	Case 2 dc = 2.5 in. γ = 0.75 ρ_g = 1% Two faces
1	1.024	0.196	0.250	0.68	1.50	
1	0.909	0.353	0.250	0.60	1.51	
1	0.815	0.457	0.250	0.55	1.49	
1	0.731	0.527	0.250	0.51	1.43	
1	0.646	0.579	0.250	0.48	1.34	
					Mean = 1.46	
3	1.276	0.078	0.25	1.00	1.28	Case 3 dc = 2.5 in. γ = 0.75 ρ_g = 3% Distributed
3	1.208	0.177	0.25	1.00	1.21	
3	1.092	0.324	0.25	0.91	1.20	
3	0.979	0.441	0.25	0.82	1.19	
3	0.870	0.529	0.25	0.75	1.16	
3	0.785	0.579	0.25	0.71	1.10	
					Mean = 1.19	
3	1.416	0.079	0.250	1.00	1.42	Case 4 dc = 2.5 in. γ = 0.75 ρ_g = 3% Two faces
3	1.343	0.187	0.250	1.00	1.34	
3	1.223	0.344	0.250	0.90	1.37	
3	1.105	0.469	0.251	0.80	1.39	
3	0.990	0.566	0.251	0.72	1.37	
3	0.899	0.622	0.251	0.68	1.32	
					Mean = 1.37	

Note: dc = cover to center of longitudinal reinforcement. Refer to Fig. 2(b) and 3(c). Mean ratio indicates ratio of analytical-to-proposed.

quite conservative. For simplicity, the effect of cross-sectional shape on effective flexural stiffness can be disregarded, provided I_g is calculated for the cross section being used.

As can be seen from Table 3 to 5, the effects of concrete cover, reinforcement distribution, and cross-sectional shape are not significant. Equation (3), which is originally derived for a square column with distributed reinforcement (Fig. 2(b)), is quite reasonable and practical and can be recommended for simplicity.

Before comparing the proposed equation (Eq. (3)) with test results, as is done in a companion paper (Khuntia and Ghosh 2004), it is appropriate to discuss the current ACI code provisions concerning *EI* of columns. In addition, the *EI* of reinforced concrete beams needs to be properly formulated, as some of the frame tests reported in the comparisons included flexural members.

FLEXURAL STIFFNESS OF REINFORCED CONCRETE BEAMS

Parametric study

A parametric study was undertaken to investigate the influence of various parameters on the effective *EI* of a reinforced

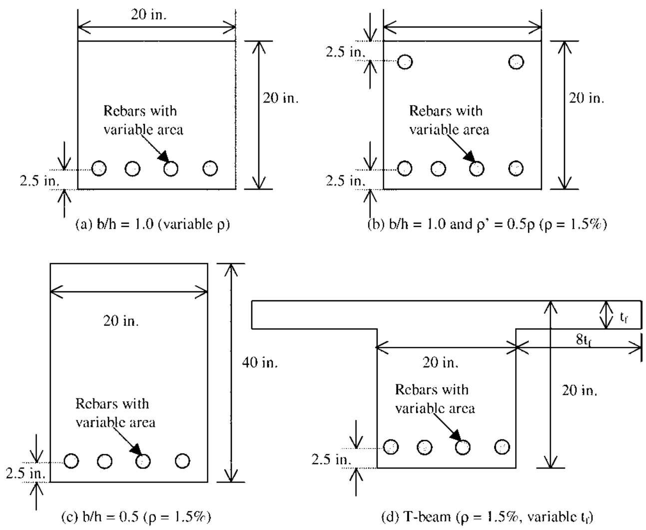

Fig. 8—Cross section of typical reinforced concrete beams considered in parametric study.

Table 5—Effective *EI* of circular reinforced concrete columns (f_c' = 4000 psi)

	Analysis			Proposed Eq. (3)		
ρ_g, % [1]	EI_e/E_cI_g [2]	P_u/P_o [3]	e/h [4]	EI_e/E_cI_g [5]	Ratio [6] = [2]/[5]	Various cases
1	0.96	0.18	0.203	0.74	1.30	Case 1 dc = 2.5 in. γ = 0.75 ρ_g = 1%
1	0.87	0.33	0.198	0.67	1.30	
1	0.77	0.46	0.192	0.61	1.27	
1	0.68	0.56	0.184	0.56	1.20	
					Mean = 1.27	
3	1.15	0.16	0.212	1.00	1.15	Case 2 dc = 2.5 in. γ = 0.75 ρ_g = 3%
3	1.05	0.30	0.208	1.00	1.05	
3	0.95	0.41	0.204	0.91	1.04	
3	0.86	0.51	0.199	0.85	1.01	
					Mean = 1.06	

Note: Effective EI_e/E_cI_g of square columns with gross reinforcement ratios of 1 and 3% (e/h = 0.2) are 1.49 and 1.21, respectively. Refer to Fig. 2(b) and 3(d). Mean ratio indicates ratio of analytical-to-proposed.

concrete beam. The primary variable considered is the tensile reinforcement ratio ρ (ranging from 0.5 to 2.5%). The secondary variables are: compressive strength f_c' (4000 and 12,000 psi), aspect ratio b/h (0.5 and 1.0), and presence of compression steel ρ' (0.0 and 0.5ρ). In addition, the influence of flanges (in the case of T-beams) on the effective *EI* is also investigated.

A 20 x 20 in. square section, with 2.5 in. cover to the center of longitudinal reinforcement, is considered as the base in the parametric study (Fig. 8(a)). However, a different reinforcement configuration with compression steel (Fig. 8(b)), a section with a different b/h ratio (Fig. 8(c)), and a flanged section (Fig. 8(d)) are also considered in the parametric study.

The flexural stiffness is calculated as the ratio of bending moment over curvature ($EI_e = M/\phi$). It should also be emphasized that the magnitude of EI_e is computed up to the yielding of the tensile reinforcement, as the value would drastically decrease after steel yielding and is of little importance for frame analysis.

The moment-curvature relationship is developed using a parabolic concrete stress-strain relationship (Eq. (1)) with peak stress at a strain of 0.002 for f_c' = 4000 psi and 0.0024 for f_c' = 12,000 psi (Fig. 2(a)). The ultimate compressive strain in concrete for all concrete strengths is taken as 0.003. It may be emphasized that the assumption of a slightly different stress-strain curve for high-strength concrete will have negligible effects on the proposed stiffness model. The yield strength of the reinforcing steel is assumed to be 60 ksi.

The effects of various parameters on the effective *EI* of reinforced concrete beams are discussed below.

Influence of tensile reinforcement ratio ρ—Figure 9 shows the moment-curvature relationships for a typical singly reinforced concrete beam with different tensile reinforcement ratios. Compressive strength of concrete equal to 4000 psi, and a cross-sectional aspect ratio (b/h) equal to 1.0 are used for plotting Fig. 9. Figure 9 shows that the effective *EI* increases with an increase in the reinforcement ratio. The reason for the increase in EI_e is that when more reinforcement is provided, the depth of the flexural cracks decreases (as more concrete depth is needed to have equilibrium of forces).

Table 6 shows the effect of reinforcement ratio on the effective *EI* of beams. In the table, analytical refers to values obtained from the theoretical moment-curvature relationships. The table also shows the variation of *EI* with compressive strain at the extreme concrete fiber and tensile strain at the level of reinforcement. Note that an increase in ε_c (Column 2) indicates an increase in applied moment; the same is true of $\varepsilon_s/\varepsilon_y$. Table 6 shows that the effective *EI* prior to the yielding of the tension reinforcement (shown in Column 5) is independent of the level of applied load (given by Column 2, 3, or 4) for steel ratios up to 1.5%. This is due to the fact that the compressive strain in the extreme concrete fiber ε_c does not exceed 0.001 before yielding of the main steel. The analysis shows, however, that for reinforcement ratios greater than 1.5%, the effective *EI* prior to the yielding of the tension reinforcement depends on the magnitude of the bending moment, that is, EI_e decreases, though not significantly, when the applied moment approaches the value corresponding to the yielding of the tension reinforcement (refer to Column 5 of Table 6 for ρ = 2.5%). This is because of higher compressive strain in the concrete (approximately 0.002) just prior to the onset of yielding of the reinforcing bars. For example, when *M* increases by 50%, the corresponding increase in ε_c (and the corresponding ϕ) is much more than 50% (for example, 70%). Therefore, an increase in *M* results in a reduction in effective *EI* for a beam in which the reinforcement ratio is quite high (more than 2%).

Influence of compression steel ratio ρ'—It has been found from analyses that the effect of compression steel on the effective *EI* of a beam is marginal, especially for beams with reinforcement ratios ρ < 1.5%. Figure 9 shows a comparison between flexural stiffnesses EI_e of beams with (Fig. 8(b)) and without (Fig. 8(a)) compression steel. As can be seen, the presence of compression steel only makes the section more ductile without any appreciable increase in the flexural stiffness of the section. For conservatism and simplicity, the effect of compression steel on the EI_e of reinforced concrete beams can be neglected.

Influence of aspect ratio b/h—Analysis shows that beams with low b/h (or b/d) ratios have higher effective *EI* than beams with larger b/h (or b/d) ratios, when both width b and reinforcement ratio ρ are the same. Note that the effective *EI* will not change for beams having the same effective depth and reinforcement ratio. Figure 10 shows the influence of aspect ratio on the EI_e of beams. Analysis shows that the

Table 6—Moment of inertia for reinforced concrete beams (f_c' = 4000 psi)

ρ_g, % [1]	ε_c [2]	$\varepsilon_c/\varepsilon_{max}$ [3]	$\varepsilon_s/\varepsilon_y$ [4]	Analytical EI_e/E_cI_g [5]	Transformed EI_e/E_cI_g [6]	Proposed Eq. (7) EI_e/E_cI_g [7]	Ratio [8] = [5]/[6]	Ratio [9] = [5]/[7]
0.5	0.00010	0.03	0.15	0.23	0.22	0.23	1.04	1.02
0.5	0.00020	0.07	0.31	0.23	0.22	0.23	1.03	1.01
0.5	0.00030	0.10	0.46	0.23	0.22	0.23	1.02	1.00
0.5	0.00040	0.13	0.60	0.23	0.22	0.23	1.02	1.00
0.5	0.00050	0.17	0.75	0.22	0.22	0.23	1.02	1.00
0.5	0.00067	0.22	1.00	0.22	0.22	0.23	1.01	0.99
1.5	0.00010	0.03	0.08	0.53	0.52	0.48	1.02	1.12
1.5	0.00020	0.07	0.16	0.53	0.52	0.48	1.02	1.12
1.5	0.00030	0.10	0.24	0.53	0.52	0.48	1.01	1.11
1.5	0.00040	0.13	0.32	0.52	0.52	0.48	1.01	1.10
1.5	0.00050	0.17	0.39	0.52	0.52	0.48	1.00	1.10
1.5	0.00100	0.33	0.74	0.50	0.52	0.48	0.97	1.06
1.5	0.00141	0.47	1.00	0.49	0.52	0.48	0.94	1.03
2.5	0.00010	0.03	0.06	0.76	0.73	0.60	1.04	1.26
2.5	0.00020	0.07	0.12	0.75	0.73	0.60	1.03	1.25
2.5	0.00030	0.10	0.17	0.75	0.73	0.60	1.02	1.24
2.5	0.00040	0.13	0.23	0.74	0.73	0.60	1.01	1.23
2.5	0.00050	0.17	0.28	0.74	0.73	0.60	1.01	1.23
2.5	0.00100	0.33	0.53	0.71	0.73	0.60	0.97	1.18
2.5	0.00150	0.50	0.74	0.67	0.73	0.60	0.92	1.12
2.5	0.00200	0.67	0.92	0.64	0.73	0.60	0.87	1.06
2.5	0.00225	0.75	1.00	0.62	0.73	0.60	0.84	1.03
						Mean =	0.99	1.10
						s.d. =	0.05	0.09

Note: ε_{max} = 0.003; $\varepsilon_s/\varepsilon_y$ = 1 indicates yielding of tension reinforcement.

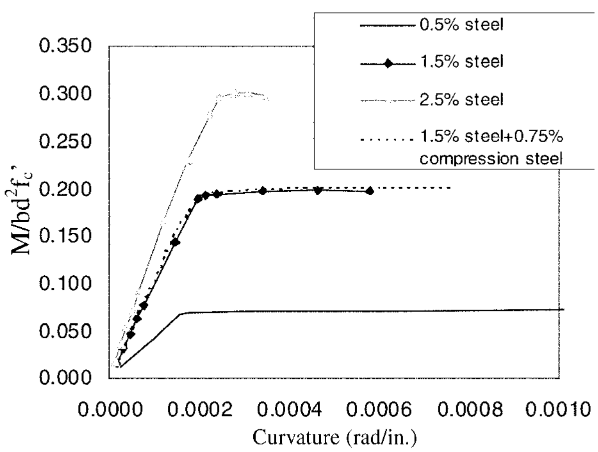

Fig. 9—Moment-curvature relationships for typical reinforced concrete beams: influence of tensile reinforcement ratio.

beam in Fig. 8(c) (b/d = 0.53) has 25% higher effective stiffness compared with the beam in Fig. 8(a) (b/d = 1.14) for a reinforcement ratio of 1.5%. Therefore, using the proposed effective EI for beams (Eq. (7), shown later), which has been derived for a beam with b/d = 1.14, would be quite conservative for most practical beams for which b/d is generally less than 1.14. For the cases where b/d is more than 1.14, however, the result may be unconservative. Analysis shows that when the b/d ratio changes from 1.14 to 2.00, there is a 25% reduction in effective EI for a beam with a reinforcement ratio of 1.5%. Equation (7) contains a term to account for this effect.

Influence of compressive strength of concrete f_c'—The effect of concrete strength on the effective EI of beams is not negligible. The reason is that when high-strength concrete is used, the depth of flexural cracks is greater (or the neutral axis depth c is smaller to maintain force equilibrium), leading to a reduction in the effective moment of inertia. The EI_e for high-strength concrete beams, however, can be larger than that for normal-strength concrete beams due to the higher E_c of the former. Table 7 shows the effective flexural stiffness of beams (shown in Fig. 8(a)) for high-strength concrete (f_c' = 12,000 psi). A comparison of Table 7 (Column 5) with Table 6 (Column 5) can be made to observe the influence of high-strength concrete on the EI_e/E_cI_g of beams. It shows lower EI_e/E_cI_g in the case of high-strength concrete, more so when the reinforcement ratio is low.

Proposed simplified equation for moment of inertia of rectangular beams

Considering the factors previously mentioned, and based on the parametric study, a simplified equation (Eq. (7)) is proposed for the effective EI of reinforced concrete beams with normal-strength concrete

$$EI_e = E_cI_g(0.10 + 25\rho)\left(1.2 - 0.2\frac{b}{d}\right) \leq 0.6E_cI_g \qquad (7)$$

where $(1.2 - 0.2b/d) \leq 1.0$. For high-strength concrete, Eq. (7) can be modified to

$$EI_e = E_cI_g(0.10 + 25\rho)\left(1.2 - 0.2\frac{b}{d}\right) \qquad (8)$$

$$\times (1.15 - 4 \times 10^{-5} f_c') \leq 0.6E_cI_g$$

where $(1.2 - 0.2b/d) \leq 1.0$.

Table 7—Moment of inertia for high-strength reinforced concrete beams (f_c' = 12,000 psi)

ρ_g, % [1]	ε_c [2]	$\varepsilon_c/\varepsilon_{max}$ [3]	$\varepsilon_s/\varepsilon_y$ [4]	Analytical EI_e/E_cI_g [5]	Transformed EI_e/E_cI_g [6]	Proposed Eq. (8) EI_e/E_cI_g [7]	Ratio [8] = [5]/[6]	Ratio [9] = [5]/[7]
0.5	0.00010	0.03	0.28	0.139	0.14	0.15	0.96	0.92
0.5	0.00020	0.07	0.56	0.138	0.14	0.15	0.96	0.92
0.5	0.00030	0.10	0.84	0.138	0.14	0.15	0.96	0.92
0.5	0.00035	0.12	0.98	0.137	0.14	0.15	0.95	0.91
1.5	0.00010	0.03	0.15	0.362	0.36	0.32	1.00	1.14
1.5	0.00020	0.07	0.31	0.360	0.36	0.32	1.00	1.13
1.5	0.00030	0.10	0.46	0.357	0.36	0.32	0.99	1.12
1.5	0.00040	0.13	0.60	0.358	0.36	0.32	0.99	1.12
1.5	0.00050	0.17	0.75	0.356	0.36	0.32	0.99	1.12
1.5	0.00067	0.22	0.98	0.355	0.36	0.32	0.99	1.12
2.5	0.00010	0.03	0.11	0.546	0.53	0.49	1.03	1.12
2.5	0.00020	0.07	0.23	0.542	0.53	0.49	1.02	1.12
2.5	0.00030	0.10	0.34	0.540	0.53	0.49	1.02	1.11
2.5	0.00040	0.13	0.45	0.538	0.53	0.49	1.02	1.11
2.5	0.00050	0.17	0.56	0.537	0.53	0.49	1.02	1.11
2.5	0.00094	0.31	0.99	0.530	0.52	0.49	1.02	1.09
						Mean =	0.99	1.07
						s.d. =	0.03	0.09

Note: ε_{max} = 0.003; $\varepsilon_s/\varepsilon_y$ = 1 indicates yielding of tension reinforcement.

Although Eq. (8) differs from Eq. (7) for f_c' more than 4000 psi, it is suggested that Eq. (8) be used for f_c' greater than 6000 psi for better accuracy.

Table 6 shows the comparison between the analytical (based on the parametric study) and the proposed EI_e values (based on Eq. (7)) for a beam with concrete strength of 4000 psi. As can be seen, the proposed Eq. (7) compares well (Column 9 of Table 6) with the results of the parametric study. The mean analytical/predicted ratio of EI_e/E_cI_g was computed to be 1.10 with a standard deviation of 0.09. It may be noted that the proposed Eq. (7) gives an EI_e of $0.25E_cI_g$ for a beam with 0.6% steel, $0.35E_cI_g$ with 1% steel, and $0.60E_cI_g$ with 2% steel.

The upper limit for EI_e of $0.6E_cI_g$ is suggested in Eq. (7) and (8) based on the analytical results. Table 6 shows that for ρ equal to 2.5%, the effective *EI* decreases with applied moment. In other words, the effective *EI* decreases significantly with increasing moments for beams with higher reinforcement ratios. An EI_e of $0.6E_cI_g$ gives a lower-bound estimate for beams with ρ > 2%. It may be noted that most practical beams have ρ between 0.75 and 1.5%.

Table 7 shows comparisons between the proposed Eq. (8) and analytical results for a beam with high-strength concrete (f_c' = 12,000 psi). The table shows that predictions by the proposed expression compare well with the analytical results.

It may be noted that the true *EI* for an entire beam is always higher than the effective *EI* of a cracked section. Therefore, it is quite reasonable to use Eq. (7) for computing the EI_e of reinforced concrete beams of all concrete strengths, for simplicity.

Comparison with transformed area method

Values given by the proposed method are also compared with the cracked moment of inertia I_{cr}, calculated using the transformed area concept (Table 6 and 7). In a simplified way, the I_{cr} by the transformed area method (for a rectangular section) can be computed as

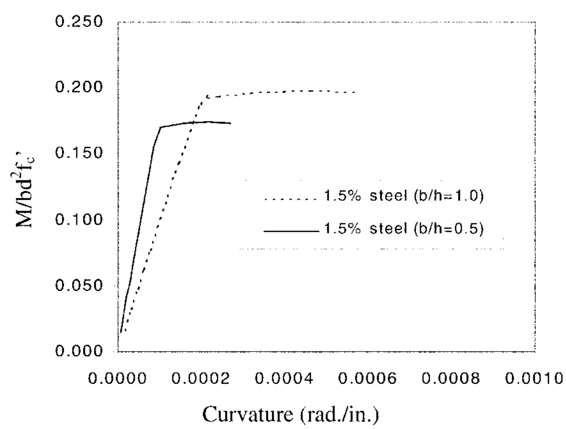

Fig. 10—Moment-curvature relationship for typical reinforced concrete beams: influence of aspect ratio b/h.

$$I_{cr} = \frac{bc^3}{3} + nA_s(d - c)^2 \quad (9)$$

where b is width; d is effective depth; c is neutral axis depth; n is modular ratio (E_s/E_c); and A_s is the area of tensile reinforcement. Table 6 and 7 show that both the proposed equations (Eq. (7) and (8)) and the transformed area method (Eq. (9)) compare well with the results of the parametric study. The proposed procedure, however, is simpler than the transformed area method because it does not require the calculation of neutral axis depth.

It needs to be emphasized that mainly the reinforcement on the tension side contributes to the flexural stiffness of beams, whereas for columns, the reinforcement over the whole section is generally effective.

Effective *EI* of T-beams

The influence of flanges on the effective *EI* of a beam is also investigated (Fig. 8(d)). In the analyses, the flange thickness-to-overall depth (t_f/h) is varied from 0.0 to 0.25. The flange

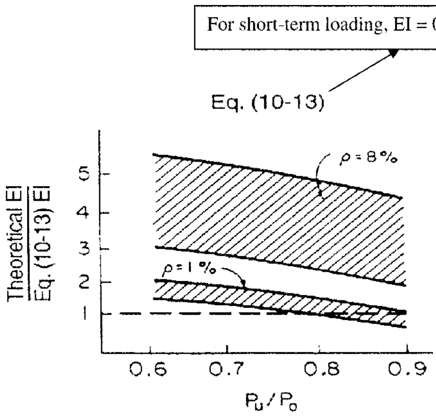

Fig. 11—Comparison of ACI 318-95, Eq. (10-13) (same as Eq. (10-12) in ACI 318-02) for EI *with* EI *values from moment-curvature diagrams (MacGregor, Breen, and Pfrang 1970).*

width is reasonably taken (based on ACI 318 Section 8.10.2) as $b_w + 16t_f$, as shown in Fig. 8(d). The reinforcement ratio (A_s/b_wd) is taken as 1.5% and the concrete strength is assumed to be 4000 psi. Table 8 shows the effect of flange thickness on the effective *EI* of a beam. The table shows that the EI_e/E_cI_g increases with an increase in flange thickness.

Mathematically, Eq. (10) reasonably represents the effective *EI* of a T-beam

$$\frac{EI_{eT}}{EI_e}\left(1 + 2\frac{t_f}{h}\right) \le 1.4 \tag{10}$$

In Eq. (10), the magnitude of EI_e of a beam without flanges can be obtained using Eq. (7). For example, if a T-beam has a t_f/h of 0.15, f_c' of 4000 psi, and ρ (= A_s/b_wd) of 1.5%, then the effective *EI* of the T-beam (using Eq. (7) and (10)) can be taken as $0.62E_cI_g$ ($I_g = b_wh^3/12$), where b_w is the width of web.

It may be noted that the effective *EI* for an inverted T-beam can be taken equal to that of a rectangular beam with a width equal to the web width because the flange in tension is ineffective.

Recommendation for effective *EI* of beam

Based on the previous discussions, it is recommended that the effective EI for rectangular beams be calculated either: a) by using the proposed simplified expression (Eq. (7)); or b) by using the transformed area concept (Eq. (9)). For T-beams with flanges in compression, a higher *EI* may be used in accordance with Eq. (10). For beams with a concrete compressive strength of more than 6000 psi, Eq. (8) may be used for better accuracy.

FLEXURAL STIFFNESS RECOMMENDATIONS OF ACI 318

The ACI 318-02 provisions (Sections 10.11 to 10.13) on effective *EI* of beams and columns are based on two significant papers: MacGregor, Breen, and Pfrang (1970) and MacGregor (1993).

In MacGregor (1993), which formed the basis of the 1995 and the 1999 ACI Code provisions on slender columns (unchanged in ACI 318-02), two sets of *EI* values for columns are recommended. The first set represents a lower-bound *EI* for individual columns, based on recommendations by MacGregor, Breen, and Pfrang (1970), as it is felt that the use of lower-bound values would be proper. For frame analysis, however, MacGregor (1993) recommended a second set of higher *EI* values because frame analysis involves all the members of a structure. Based on extensive studies by other investigators, MacGregor (1993) recommended that a reasonable estimate of *EI* for second- or first-order elastic analyses be based on the ACI value of E_c (Section 8.5.1) and $I = 0.4I_g$ for beams and $0.8I_g$ for columns. These values were originally suggested by MacGregor and Hage (1977). Using a stiffness reduction factor of 0.875 for frame analysis (considering the condition just prior to the attainment of strength), MacGregor (1993) suggested an effective *I* of $0.35I_g$ for beams and $0.70I_g$ for columns. Under service loads, however, he recommended an increase in the aforementioned values to $1.0I_g$ and $0.5I_g$ for columns and beams, respectively.

Table 8—Effective *EI* of reinforced concrete T-beams (f_c' = 4000 psi; ρ = 1.5%)

Ratio t_f/h [1]	Analytical EI_e/E_cI_g [2]	Magnification over rectangular beam [3]	Magnification per proposed Eq. (10) [4]	Ratio [5] = [3]/[4]
0.00	0.52	1.00	1.00	1.00
0.05	0.57	1.10	1.10	1.00
0.10	0.64	1.23	1.20	1.03
0.15	0.69	1.33	1.30	1.02
0.20	0.71	1.37	1.40	0.98
0.25	0.73	1.40	1.40	1.00
				Mean = 1.00
				s.d. = 0.02

MacGregor, Breen, and Pfrang (1970) reported findings that formed the basis of most of the ACI 318 slender column design provisions since ACI 318-71 until present day. They recommended the use of an effective *EI* for individual columns, without sustained loading, equal to $0.4E_cI_g$ (or $0.2E_cI_g + E_sI_s$) for computation of moment magnification factors. This low value of *EI* is used: a) to calculate δ_{ns} for nonsway frames per Section 10.12.3; and b) to calculate δ_s for sway frames per Section 10.13.4.3 (moment-magnifier method). Their recommendation of lower-bound *EI* (= $0.4E_cI_g$) is based on a reinforcement ratio of 1% and an axial load-to-pure axial load strength ratio P_u/P_o of more than 0.85 (Fig. 11).

Figure 11 (reproduced from the paper by MacGregor, Breen, and Pfrang [1970]) is based on an extensive theoretical study on moment-curvature relationships for columns with various reinforcement and axial load ratios. It is interesting to note that the use of a P_u/P_o ratio of 0.90 and an *e*/*h* ratio of 0.1 with ρ_g of 1% in the proposed Eq. (3) gives an *EI* of $0.47E_cI_g$, a value quite comparable to the $0.4E_cI_g$ of the ACI Code (Eq. (10-12)). In actual practice, however, the reinforcement ratio for a column is generally more than 1% (approximately 2%, for example), and, more importantly, the axial load ratio P_u/P_o is not permitted to exceed 0.56 (= 0.8 × 0.7) for tied columns or 0.64 (= 0.85 × 0.75) for columns with spiral reinforcement (using ϕ-values from Appendix C of ACI 318-02). In fact, Fig. 11 shows that for a reinforcement ratio of 1% and a P_u/P_o of 0.6 (both being the worst possible cases), the magnitude of effective *EI* would vary between 0.6 and $0.8E_cI_g$. Therefore, the lower limit of *EI* based on Fig. 11 should not be less than $0.6E_cI_g$ for any practical column.

RECOMMENDATION FOR FRAME ANALYSIS AND CONCLUSIONS

Based on results of the analytical study and their comparison with the results of existing experimental research reported in the companion paper (Khuntia and Ghosh 2004), the

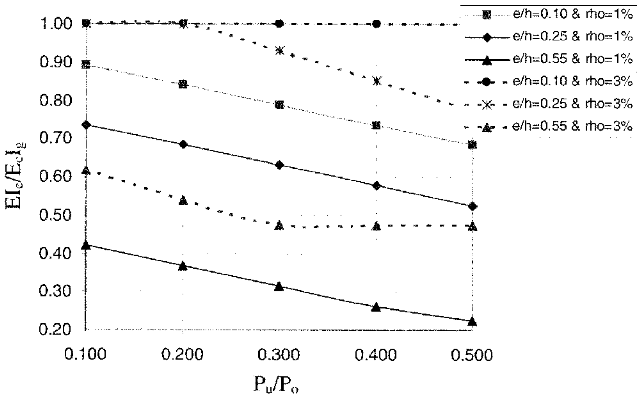

Fig. 12—Influence of axial load and eccentricity ratio on flexural stiffness of columns.

following recommendations are made concerning the effective *EI* of beams and columns to be used in the lateral analysis of frames in general and of frames including slender columns in particular:

In frame analysis (both first- and second-order elastic), it is recommended to initially assume beam $EI = 0.35E_cI_g$ (which occurs for a beam with a ρ of 1% per Eq. (7)) and column $EI = 0.70E_cI_g$ (which occurs with $\rho_g = 1.5\%$, $e/h = 0.20$ and $P_u/P_o = 0.40$ per Eq. (3)). On completion of lateral analysis, however, the effective *EI* for beams and columns need to be recalculated using Eq. (7) and (3), respectively. Note that depending on the magnitude of e/h (or M_u/P_uh), the EI_e value for columns will change. If the final EI_e values are different from the initially assumed values by more than 15% (that is, if the column EI_e is not within the range of 0.6 to $0.8E_cI_g$), it is recommended to perform the analysis again using the revised EI_e. Figure 12 shows the effective flexural stiffness of columns, which varies depending on the reinforcement, axial load, and eccentricity ratio. It shows that the assumption of $EI_e = 0.7E_cI_g$ is quite reasonable for most cases. The assumption may not be appropriate for some cases, however, especially when the reinforcement ratio is low and the eccentricity ratio is high.

For additional recommendations, readers should refer to the companion paper by Khuntia and Ghosh (2004).

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NOTATION

A_g	=	gross cross-sectional area, in.^{2}
A_{st}	=	gross steel area in column, in.^{2}
b	=	width of member, in.
b_w	=	width of web of T-beams, in.
c	=	depth of neutral axis, in.
d	=	distance form extreme compression fiber to centroid of tension reinforcement in flexural member
E_c	=	modulus of elasticity of concrete, ksi
E_s	=	modulus of elasticity of reinforcing steel, ksi
EI	=	flexural stiffness of member or cross section, in.^{2}-lb
e	=	eccentricity of axial load, in.
e/h	=	eccentricity ratio = M/Ph = (M_u/P_uh in context of strength design)
f_c	=	compressive stress in concrete at a strain of ε_c, psi
f_c'	=	specified compressive strength of concrete, psi
f_y	=	yield strength of reinforcement, ksi
h	=	overall depth of member, in.
I	=	moment of inertia of cross section, in.^{4}
I_{beam}	=	effective moment of inertia of flexural member, in.^{4}
I_{cr}	=	moment of inertia of cracked cross section of flexural member, calculated using transformed area concept
I_e	=	effective moment of inertia of cross section, in.^{4}
I_{eT}	=	effective moment of inertia of T-beam, in.^{4}
I_g	=	moment of inertia of gross concrete section about centroidal axis, neglecting reinforcement, in.^{4}
I_s	=	moment of inertia of reinforcing steel about centroidal axis, in.^{4}
M	=	bending moment, in.-lb = M_u in context of strength design
M_n	=	nominal flexural strength, in.-lb
M_u	=	factored moment or required moment strength at section, in.-lb
n	=	modular ratio = E_s/E_c
P	=	axial load, kips = P_u in context of strength design
P_n	=	nominal axial load strength, kips
P_o	=	nominal axial load strength at zero eccentricity, kips
P_u	=	factored axial load or required axial load strength, kips
P_u/P_o	=	axial load ratio
t_f	=	flange thickness of T-beams, in.
V	=	lateral force, kips
w_c	=	unit weight of concrete, lb/ft^3
Δ	=	story drift, in.
δ	=	deflection of compression member relative to chord joining ends of column in deflected frame, in.
ε_c	=	compressive strain in concrete, in./in.
ε_{max}	=	maximum compressive strain in concrete, in./in.
ε_o	=	compressive strain in concrete at peak stress, in./in.
ε_s	=	tensile strain in steel, in./in.
ε_y	=	yield strain in steel, in./in.
ϕ	=	strength reduction factor
	=	curvature at section, rad./in.
γ	=	ratio of distance between centerlines of outermost bars to overall dimension of section
ρ	=	tensile reinforcement ratio in flexural member, A_s/bd, %
ρ_g	=	gross reinforcement ratio in compression member, A_{st}/A_g, %

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