ROTATION COMPATIBILITY IN THE LIMIT DESIGN OF REINFORCED CONCRETE CONTINUOUS BEAMS

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SYNOPSIS

The object of the paper is to provide a simple, rational technique to check the rotation compatibility of plastic hinges in limit designed reinforced concrete continuous beams proportioned basing on optimum considerations.

The relationship between the plastic adaptability and the rotation compatibility is outlined, expressing conveniently both the inelastic rotations and the rotation capacities of critical sections. It is concluded that the compatibility requirement implies only limited adaptability to be used in the design of concrete structures. Since a similar conclusion can be derived with regard to the serviceability conditions of limit designed structures, adoption of convenient upper bounds for the redistribution factors (or lower bounds for the yield safety parameters) of critical sections will implicitly provide adequate solutions for ultimate safety, compatibility, and serviceability as well.

From the practical viewpoint, the significant result follows that for given (1) properties of materials, (2) loading conditions, and (3) amount of accepted redistribution, the rotation compatibility condition to an upper limitation of the steel percentages at critical sections.

INTRODUCTION

Three fundamental conditions are specific to the limit design of reinforced concrete structures, namely: (1) Limit equilibrium: (2) rotation compatibility; and (3) serviceability.

The first conditions postulates the existence of one or more collapse mechanisms. This configuration can theoretically be reached owing to the plastic adaptability of the structure.

The second condition implies that all plastic hinges necessary for a structural collapse may actually occur, without premature local fracture of the concrete. Rotation compatibility is thus the property of plastic hinges through which plastic adaptability can become effective, conforming to the real properties of steel and concrete.

The third condition requires a reasonable yield safety of critical sections, on which the magnitude of the crack openings and deflections at working loads are finally depending.

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It has elsewhere been shown [1][2] that limit design methods consider two of these three conditions initially, the third being object of subsequent check. A first approach is to place the main emphasis on conditions (1) and (2), as in references [2], [3], [4], [5].

Recently, however, quite a different approach was proposed [1] based on the simultaneous fulfilment of conditions (1) and (3). Two procedures were suggested and purposely denoted as "optimum limit design" (OLD) procedures [6], [7].

However, since no consideration was given to the rotation compatibility, the problem arised whether or not the OLD solutions as indicated by these procedures, were conforming to condition (2).

The object of the present paper is to suggest adequate means for checking the rotation compatibility of limit designed R. C. continuous beams supplementing the OLD solutions, as given in references [1], [6], [7].

Suitable expressions for both the inelastic rotations and the rotation capacities of plastic hinges are derived. Therefrom it is concluded that the compatibility condition reduces to an upper limitation of the redistribution factors (or to the corresponding lower limitation of the yield safety parameters) of the critical sections. Since the serviceability criteria adopted in the OLD procedures resulted in the same conclusion, it follows that a design for limited plastic adaptability will be satisfactory in relation to all the three mentioned basic conditions.

As will be seen, the optimum limit design implying a given amount of moment redistribution, the compatibility condition leads to an upper limitation of the steel percentages at critical sections. The main result of the present investigation follows: A safe and serviceable plastic moment distribution is also compatible, provided the reinforcement of critical sections does not exceed some definite upper bounds.

For illustrating purposes these are presented through diagrams relating p, r, and x parameters, for current grades of steel and concrete, derived under the usual assumptions concerning the inelastic properties of R. C. members in flexure. Therefrom not only specific figures for individual designs may be deduced, but the effect of such parameters as the amount of compression reinforcement and the quality of concrete becomes apparent.

Though adoption of more refined assumptions on the inelastic behavior of R. C. sections in bending [8] is highly desirable and is expected to bring more accuracy in analytical and experimental work, the background of the present treatment of the compatibility condition will remain unaffected.

Notation.—The symbols adopted for use in this paper are defined where they first appear and are arranged alphabetically in Appendix I.

PREVIOUS INVESTIGATIONS

Some authors use the compatibility equations as a general basis of ultimate load design. A. L. L. Baker extended the well known equations of the elastic structural analysis obtaining the expressions of the plastic rotations at critical sections [2], [3], [9]. Guyon [4], Macchi [5] and Jain [10] also proposed ultimate load methods of analysis for continuous beams, based on a limited deformation capacity of reinforced concrete sections.

[2] Numerals in brackets refer to corresponding items in the Appendix II.

Values of the plastic rotations at typical critical sections are given by Sawyer [11] together with an elastic-plastic procedure for analysing beams and frames taking account of the available deformation capacity of R. C. plastic hinges.

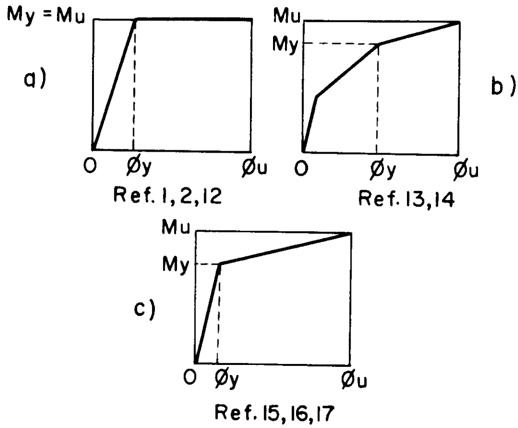


FIG. 1

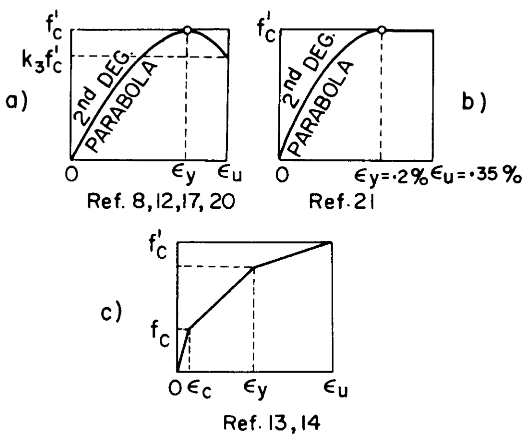


FIG. 2

Generally, these methods are developed by adopting various idealizations of the moment-curvature relationship, some of which are indicated in Fig. 1. References to the original reports [12] . . . [17] are given by the numbers in brackets on Fig. 1.

An interesting way of checking the rotation compatibility is suggested in a paper by Carneiro [18], by an upper limitation of the depth to length ratio of members.

From the structural viewpoint the most general statement of the problem seems to be that implied in a paper by Hangan [19]. Such a formulation will be developed in the next section of the present paper.

Turning now to the sectional aspect of the problem the rotation capacity of plastic hinges depends essentially on the inelastic properties of reinforced concrete sections, i. e., stress-strain and moment-curvature relationships, values of moments, and flexural rigidities of critical sections from their first yield to collapse.

The most usual idealizations of the stress-strain relationship are indicated in Fig. 2, along with some references where they are adopted.

Theoretical contributions to the determination of the rotation capacity of plastic hinges are due to A. L. L. Baker [2], Ernst [12], Chan [13], [29], Sawyer [11], [15], Miraz [22], Yamada [24], etc.

Experimental work on the problem has been reported by A. L. L. Baker [3], [2], Chan [13], Sawyer [15], Wright and Berwanger [17], Petcu and the writer [25], etc.

An overall evaluation of the available literature justifies the following brief remarks:

1. Rotation compatibility is essentially a problem of analysis since it implies that dimensions of members and properties of materials are known;
2. usual methods for checking the compatibility conditions are step-by-step procedures, starting from the limit stage of the structure;
3. determination of the inelastic rotations of critical sections is a tedious operation requiring time, ability, and limit design experience;
4. assessment of the rotation capacities of plastic hinges is subject to large approximations, in view of the uncertainties concerning the actual properties of steel and concrete in the inelastic range and of the empirical nature of the adopted safe-limiting parameters.

These considerations lead the writer to the approach below, which without eliminating the inherent approximations specific to the problem, presents, at least, the triple advantage of safety, simplicity and of handling a single parameter of arbitrary choice. This approach has initially been suggested in a discussion to the Report on Ultimate Load Design by the ICE Research Committee [28].

GENERAL STATEMENT OF THE COMPATIBILITY CONDITION

Consider a redundant reinforced concrete structure failing when the necessary number of critical sections become plastic hinges. Let Q_i be the load corresponding to the formation of a plastic hinge at section 1. (Q_i may be either a proportional load or a combined load, consisting of a standing and a proportionally increasing load, i. e., $Q_i = G + P_i$.)

The critical sections being indexed in the order of their successive yielding, the first plastic hinge appears at a load Q_1, while the collapse occurs at Q_u.

For a definite loading level $Q > Q_1$ the inelastic rotation of the plastic hinge is denoted as θ_i. (For $Q < Q_1$ by definition $\theta_i = 0$).

Now let θ_{pi} be the rotation capacity of a section i, derived from the consideration of the actual properties of materials and the effective stress distribution at a given loading level.

If for any $1 \leq i \leq u$ the effective plastic rotation is less than the rotation capacity, i. e.,

$$\theta_i \leq \theta_{pi} \quad \text{. . . . .} \quad (1a)$$

a "complete collapse" (or full redistribution) occurs for the ultimate load Q_u, the solution fulfilling the condition of deformation compatibility.

If for at least one value of i the effective rotation at plastic hinge i exceeds the rotation capacity of the section, i. e.,

$$\theta_i > 0_{pi} \quad \text{. . . . .} \quad (1b)$$

a local fracture of concrete occurs at i. In this case an "incomplete" or "partial collapse" (or partial redistribution) is recorded for an ultimate load Q_u, so that

$$Q_i \leq Q_u < Q_{i+1} \quad \text{. . . . .} \quad (2)$$

The meaning of relation 1a expressing the general condition of compatibility for the plastic solution of a redundant reinforced concrete structure is as follows: The inelastic rotation of any plastic hinge should remain less than the available rotation capacity at each loading level, i. e., from the occurrence of the first to the last plastic hinge of a structure.

Accordingly, the following steps are involved in checking the compatibility requirements:

1. Determine the collapse mechanism and the ultimate load Q_u;
2. determine the loads Q_i corresponding to the progressive formation of plastic hinges at sections i;
3. derive the effective plastic rotations θ_i $(1 \leq i \leq u)$;
4. evaluate for all critical sections the available rotation capacities θ_{pi};
5. check the condition 1a for all the plastic hinges at each loading level Q_i:

The actual collapse load is the largest value of Q_i for which relation 1a is satisfied at the limit $\theta_i = \theta_{pi}$.

The exact evaluation of Q_i and θ_{pi} at all stages of loading is only possible by a historical analysis of structures, based on elastic considerations.

A systematic procedure for the step-by-step determination of the loads and rotations (Q_i and θ_i) as the plastic hinges appear throughout the structure has been presented by Hangan [19]. The procedure provides a means to check the rotation compatibility in its general form, according to relation 1a.

The general procedure is difficult enough, even for application to the simplest cases. Its complexity becomes excessive with the increased number of critical sections involved in the historical analysis of highly redundant structures.

The amount of labor necessary to check the condition of compatibility becomes still larger when more possible loading schemes are to be considered.

Indeed, in this case the maximum rotations of individual plastic hinges correspond to distinct arrangements of loads. Therefore a set of historical analyses of the structure is necessary, equal to the number of independent loading schemes required.

However, in many cases (and particularly for structures failing as beam-mechanisms) the labor implied by such a treatment is unnecessary. Since beams collapse with at most three plastic hinges, it will be sufficient to check the rotation capacity of the first plastic hinge to form, as it will generally exhibit the largest plastic rotation.

Below a simple approach will be presented largely simplifying the study of the compatibility condition for limit designed reinforced concrete continuous beams with equal spans.

INELASTIC ROTATIONS OF CRITICAL SECTIONS

Preliminary Remarks.—Consider a typical five-span continuous reinforced concrete beam, the inelastic rotations being required for each critical section.

For reasons which will become apparent later the discussion is restricted to support critical sections alone.

In order that a given support section i to become the first plastic hinge the live load must be applied as shown in Fig. 3(a), b. Note that for any critical section the spans adjacent to the support considered are loaded.

Under this particular loading arrangement the supports j adjacent to the first plastic hinge i will behave in one of the three modes indicated in Fig. 4:

(a) As ideal hinges, the corresponding moments being zero (typical for outer spans of continuous beams with free ends);

(b) as partially-fixed ends, the moments being less than the fixed-end moments (typical for inner spans of continuous beams); and

(c) as fixed-ends (typical for outer spans of fixed-ended continuous beams).

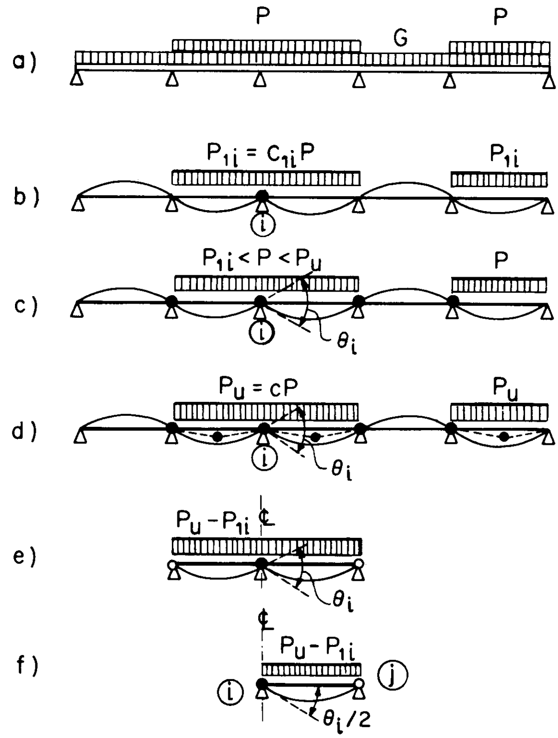
Of the three cases, the maximum inelastic rotation at plastic hinges will obviously be given by case (a). Turning again to Fig. 3, it will be noted that a larger hinge discontinuity than in reality is recorded if it is supposed that ideal hinges are inserted at the supports adjacent to the considered plastic hinges. It will, therefore, be sufficient to investigate the two-span beam extracted from the given beam, as shown in Fig. 3 (e).

But since under the above assumptions the loading and the support conditions are symmetrical it follows that study of a propped cantilever instead of the actual beam leads to conservative values of the inelastic rotations of plastic hinges.

Inelastic Rotations of Support Critical Sections.—Consider then a propped cantilever, the fixed-end of which becomes a plastic hinge i at a total load of $Q_{li} = G + P_{li}$. The inelastic rotation of the plastic hinge is the slope at point i for the simple supported beam i-k when the load is increased from $Q_{li} = G + P_{li}$ to $Q_u = G + P_u$, i. e., for $Q = Q_u - Q_{li}$ [Fig. 5(a)].

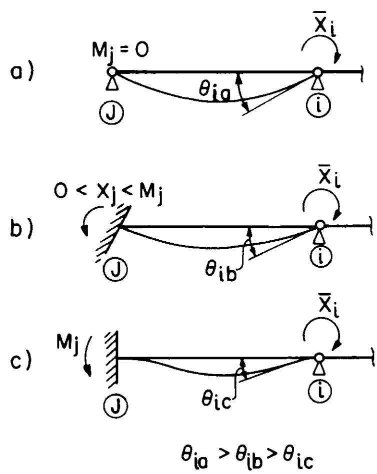
Assuming, initially, a homogeneous beam with a constant flexural rigidity, R_o, the slope at point i can be written as

$$\theta_i = \frac{1}{L} \int_o^L \frac{M_o}{R_o} (L-z)\, dz = m_i\, L/6R_o \quad \ldots\ldots\ldots (3)$$



a)
P
G
P
b)
P1i = C1i P
P1i
i
c)
P1i < P < Pu
P
i
θi
d)
Pu = cP
Pu
i
θi
e)
Pu - P1i
θi
f)
Pu - P1i
i
j
θi/2

FIG. 3



a)
Mj = 0
X̄i
θia
J
i
b)
0 < Xj < Mj
X̄i
J
θib
i
c)
Mj
X̄i
J
θic
i
θia > θib > θic

FIG. 4

in which
$$m_i = \frac{6R_o}{L^2} \int_o^L M_o (L-z)\, dz = \alpha QL \qquad (4)$$

α being a constant depending on the loading scheme, given in current handbooks.

From Eqs. 3 and 4 it follows

$$\theta_i = \frac{\alpha L^2}{6 R_o} (Q_u - Q_{1i}) \qquad (5a)$$

or
$$\theta_i = \frac{\alpha Q_{1i} L^2}{6R_o} (r_i - 1) \qquad (5b)$$

$r_i = Q_u/Q_{1i}$ being by definition the redistribution factor of the beam.

Conversely, the plastic moment being reached at i the moment at this section becomes

$$M_i = \overline{X}_i = m_i/2 = \alpha Q_{1i} L/2 \qquad (6)$$

Eliminating the value of Q_{1i} from Eqs. 5b and 6 the inelastic rotation θ_i, is found to be

$$\theta_i = \frac{\overline{X}_i L}{3R_o} (r_i - 1) \qquad (7)$$

Eq. 7 shows that the inelastic rotation of a critical section is independent of the loading scheme. It becomes larger the smaller the flexural rigidity is and the larger the redistribution factor and the plastic moment (the steel percentage of the plastic hinge) are. As expected, it is seen that θ_i vanishes when the redistribution factor tends to unity, i. e., all the plastic hinges appear simultaneously.

Eq. 7 was deduced assuming that the flexural rigidity, R_0, is constant along the beam. Because when a plastic hinge forms at the critical section, i, generally the beam is cracked both at supports and in span, the flexural rigidity is far from having a constant value. For analytical purposes the simple assumption will be accepted [14] that rigidities are constant between the point of contraflexure, the minimum values of corresponding zones being assessed thoughout [Fig. 5(c)].

Under these circumstances the slope at i becomes [27]

$$\theta_i' = \frac{1}{L} \int_o^L \frac{M_o}{R} (L-z)\, dz = \frac{s_i m_i L}{6R_o} \qquad (8a)$$

or
$$\theta_i' = \frac{s_i \alpha Q_1' L^2}{6R_o} (r_i' - 1) \qquad (8b)$$

in which primed symbols have the same meaning as before, but refer to the beam in which the variable rigidity is taken into account, s_i = an elastic constant depending on the distribution of rigidities, the ratio of span to support rigidities, (n), and the loading scheme. Thus, $s_i = 1 - (1-n)\, F_i$, and the plastic moment in this case has the form [27]

$$M_i' = \overline{X}_i' = \frac{s_i m_i}{2c_i} = \frac{s_i}{2c_i} \alpha Q_{1i}' L \quad (9)$$

Elimination of Q_{1i}' between Eqs. 8b and 9 yields

$$\theta_i = \frac{c_i \overline{X}_i' L}{3R_o} \cdot (r_i' - 1) \quad (10a)$$

Now, since from Eqs. 6 and 9 $\overline{X}_i'/\overline{X}_i = s_i/c_i \cdot Q_{1i}'/Q_{1i} > 1$ follows, and Eq. 10a may be rewritten finally as

$$\theta_i' = \frac{s_i \overline{X}_i L}{3R_o} (r_1 - Q_{1i}'/Q_{1i}) \quad (10b)$$

It is seen that an expression similar to Eq. 7 has been obtained, the influence of the variable flexural rigidity being reflected by the elastic constant, s_i, and the yield load Q_{1i}' instead of Q_{1i}.

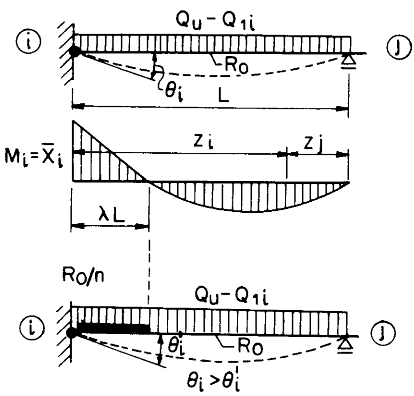


FIG. 5

Since, by definition, s_i is a dimensionless factor less than unity and Q_{1i}' is larger than Q_{1i}, by comparing Eqs. 7 and 10b, $\theta_i' < \theta_i$ follows.

Thus, consideration of a constant rigidity along the beam is a safe assumption, yielding larger inelastic rotations than would result from the consideration of the variable rigidity.

As to the actual value of R_o, adoption of the support rigidity when yield just occurs is a conservative value for practical purposes.

ROTATION CAPACITY OF REINFORCED CONCRETE PLASTIC HINGES

General Expressions of the Rotation Capacity.—The rotation capacity of a plastic hinge may be expressed as the total rotation accumulated along a short zone l_p, where yield has spread near the support under consideration.

Referring to Fig. 6, the rotation capacity is given by

$$\theta_p = \int_0^{l_p} (\phi_Z - \phi_y)\, dz = A_\phi = \beta\, l_p \phi_p \quad .\ .\ .\ .\ (11a)$$

in which $\phi = M/R$, M, and $R = EI$ are curvatures, bending moments, and

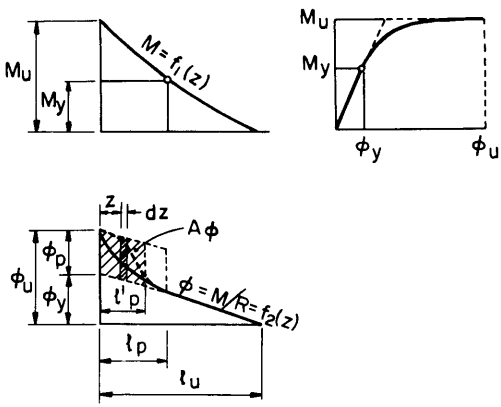


FIG. 6

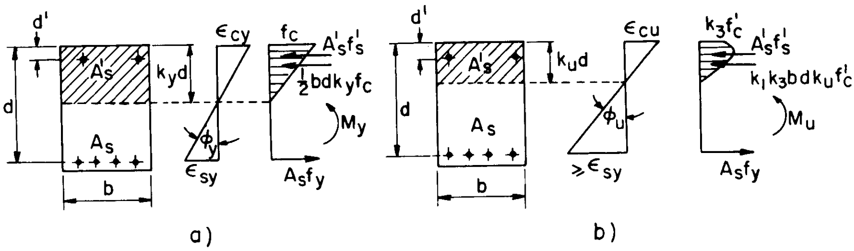


FIG. 7

flexural rigidities, respectively. Below subscripts u and y refer to ultimate and yield load conditions, while indices c and s refer to concrete and steel, respectively.

From Eq. 11a it is seen that the rotation capacity of a plastic hinge can be evaluated by the area of plastic curvatures A_ϕ (dashed zone in Fig. 6). The dimensionless factor $\beta < 1$ is a shape factor of the curvature diagram near support, which may also be regarded as a reduction factor of the plastic hinge length l_p, so that $\beta l_p = l'_p$. In this case, the rotation capacity equals the product of the maximum plastic curvature and the reduced hinge length, ie.,

$$\theta_p = l'_p \, \phi_p \quad \text{(11b)}$$

Assuming the conservation of plane sections to be valid to failure, the curvatures of the support critical section at yield and ultimate loads are, respectively

$$\phi_y = \frac{\epsilon_{cy}}{k_y d} \text{ and } \phi_u = \frac{\epsilon_{cu}}{k_u d} \quad \text{(12)}$$

since $\theta_p = \phi_u - \phi_y$ and denoting $l_p = \lambda L$ Eq. 11a becomes

$$\theta_p = \beta \, l_p \, (\phi_u - \phi_y) = \beta \lambda \left(\frac{\epsilon_{cu}}{k_u} - \frac{\epsilon_{cy}}{k_y} \right) \frac{L}{d} \quad \text{(13a)}$$

An alternative relation, due to Chan [13] may be found noting that $\phi_u = M_u/R_u$ and $\phi_y = M_u/R_y$: Thus,

$$\theta_p = \beta \, l_p \, M_u \left(\frac{1}{R_u} - \frac{1}{R_y} \right) \quad \text{(13b)}$$

A simplified expression for θ_p together with safe-limiting values involved is recommended by A. L. L. Baker [2], [3], under the conservative assumption $k_y \doteq k_u$ in Eq. 13a: Thus

$$\theta_p = \frac{\epsilon_{cu} - \epsilon_{cy}}{k_u d} \, l'_p \quad \text{(13c)}$$

Inelastic Properties of Reinforced Concrete Sections.—Eqs. 12 and 13 show that in evaluating the rotation capacity of a plastic hinge θ_p, values of ϕ, M, R, ϵ_c, ϵ_s, and k may be needed at yield and at ultimate load.

For the purpose of illustrating the basic aspects of the rotation compatibility the inelastic behavior of R. C. members will be supposed according to the commonly accepted theories.

Below rectangular sections reinforced in both tension and compression are dealt with. It is assumed $E_s = 30.10^6$ psi, $E_c = 1{,}800.000 + 460 f'_c$ psi, and $n = E_s/E_c$.

At yield, i. e., when the tension steel just reaches the yield point, the following well known relations can be written [refer to Fig. 7 (a)]:

$$k_y = \sqrt{n^2(p + p')^2 + 2n(p + p' \, \delta')} - n\,(p + p') \quad \text{(14)}$$

$$\epsilon_{cy} = \frac{f_y}{E_s} \cdot \frac{k_y}{1-k_y} \quad \text{(15)}$$

$$M_y = A_s f_y d \left(1 - \frac{k_y}{3}\right) + A'_s f_s d \left(\frac{ky}{3-\delta'}\right) \quad \text{(16)}$$

$$\phi_y = \frac{\epsilon_{cy}}{k_y d} \quad \text{(17)}$$

$$R_y = \frac{M_y}{\phi_y} \qquad (18)$$

At ultimate, i. e., when the concrete crushes, the ultimate strain is assumed to have a constant value, irrespective of the concrete grade and the steel percentages: $\epsilon_{cu} = 0.35\%$.

Referring to Fig. 7 (b), the general expression giving the depth of the compressive zone at ultimate load is

$$k_u = \frac{1}{k_1 k_3}\left(p\frac{f_y}{f'_c} - p'\frac{f'_s}{f'_c}\right) \qquad (19)$$

The following particular situations may occur:

1. $q \leq \delta'/k_1k_3$ (compressive zone depth is less than or equal to the position of compressive reinforcement: $k_u d \leq d'$). Section behaves as reinforced in tension only. Hence,

$$k_u = \frac{q}{k_1 k_3} = \frac{pf_y}{k_1 k_3 f'_c} \qquad (20a)$$

2. $\delta'/k_1k_3 < q \leq k_1k_3/2 \,.\, 1 + \delta'/1 - \mu$: In this case the axis lies below the compressive reinforcement, but the compressive reinforcement does not yield once with the tensile reinforcement. Therefore, $k_u d > d'$, $f_s = f_y$, and $f'_s < f_y$.

The position of neutral axis is defined by

$$k_u = \frac{q(1+\mu)}{2k_1k_3} + \frac{1}{2} - \sqrt{\left[\frac{q(1+\mu)}{2k_1k_3} + \frac{1}{2}\right]^2 - \frac{q(1+\mu\delta')}{k_1k_3}} \qquad (20b)$$

3. $k_1k_3/2 \,.\, 1 + \delta'/1 - \mu < q \leq 1/k_1k_3\,(1-\mu) \,.\, \epsilon_{cu}/\epsilon_{cu} + \epsilon_{sy}$: Both tension and compression steel yield ($k_u d > d'$ and $f_s = f'_s = f_y$), the depth of the compressive zone being given by

$$k_u = \frac{1-\mu}{k_1k_3}\,q = \frac{1-\mu}{k_1k_3}\,p\frac{f_y}{f'_c} \qquad (20c)$$

4. $q > \dfrac{1}{k_1k_3\,(1-\mu)} \cdot \dfrac{\epsilon_{cu}}{\epsilon_{cu} + \epsilon_{sy}}$: Balanced reinforcement: concrete crushes simultaneously with yield of compression and tension steel.

$$k_u = \frac{\epsilon_{cu}}{\epsilon_{cu} + \epsilon_{sy}} \qquad (20d)$$

The situation in which concrete crushes before steel yields should not be considered in limit design.

The ultimate moment may be expressed by

$$M_u = k_1k_3f'_ck_ubd^2(1-k_2k_u) + A'_sf'_sd(1-\delta') \qquad (21)$$

Therefrom

$$\phi_u = \epsilon_{cu}/k_u\, d \;\ldots\ldots\ldots\ldots\ldots\ldots (22)$$

$$R_u = M_u/\phi_u \;\ldots\ldots\ldots\ldots\ldots\ldots (23)$$

THE CONDITION OF ROTATION COMPATIBILITY

Again consider Eqs. 7 and 13, the former being rewritten taking into account that $\overline{X}_i/R_o = \phi_y = f_y/d(1-k_y)E_s = \epsilon_{cy}/k_y d$:

$$\theta_i = \frac{r-1}{3} \cdot \frac{\epsilon_{cy}}{k_y} \cdot \frac{L}{d} \;\ldots\ldots\ldots\ldots (24)$$

$$\theta_{pi} = \beta\,\lambda \left(\frac{\epsilon_{cu}}{k_u} - \frac{\epsilon_{cy}}{k_y}\right)\frac{L}{d} \;\ldots\ldots\ldots\ldots (13a)$$

Expressing the general condition of compatibility (Eq. 1a) in terms of these relations it is finally found that

$$r \leqslant 1 + 3\,\beta\,\lambda\left(\frac{\epsilon_{cu}}{\epsilon_{cy}} \cdot \frac{k_y}{k_u} - 1\right) \ldots\ldots\ldots\ldots (25a)$$

or

$$r \leqslant 1 + 3\,\beta\,\lambda\,(\phi_u/\phi_u - 1) \;\ldots\ldots\ldots\ldots (25b)$$

Substituting for ϵ_{cy} the value given by Eq. 15, Eq. 25a becomes

$$r \leqslant 1 + 3\,\beta\,\lambda\left(\frac{\epsilon_{cu}}{\epsilon_{sy}} \cdot \frac{1-k_y}{k_u} - 1\right) \ldots\ldots\ldots (25c)$$

Values of k_y and k_u depend essentially on the steel percentages of critical sections. Hence, for given grades of concrete and steel relations Eqs. 25 show that the compatibility problem of limit designed beams is governed by the redistribution factor and the steel percentage of the considered plastic hinge.

A limit design solution will have compatible plastic rotations so long as the redistribution factors corresponding to the critical sections do not exceed the upper limit indicated by the righthand term of Eq. 25.

In the optimum limit design of continuous R. C. beams [1] either full or partial redistribution being involved, maximum values of r to be used in a particular problem are provided, in order that suitable limit and serviceability requirements be met simultaneously,

For such beams the simple design rule below was suggested [1], [7], [28]:

The plastic moments ($\overline{X}_i$) may be obtained from the corresponding elastic values ($M_i = a_i\,GL + b_i\,PL$), reducing the influence of the factored live loads as follows: by 10% for span critical section in all cases; by 20%, 25%, and 30% for support sections of beams with 2, 3, and 4 or more spans and with free ends; -by 15%, 20%, and 25% for support sections of beams with 2, 3, and 4 or more spans and with fixed ends.

According to this rule the plastic moments will therefore be of the form $\overline{X}_i = a_i\ GL + b_i x_i\ cPL$, the reduction factors $x_i = c_{1i}/c$ being the yield safety parameters of the sections.

Since, by definition,

$$r_i = \frac{G + cP}{G + c_{1i}P} = \frac{w/c + 1}{w/c + x_i} \qquad (26)$$

from Eqs. 25 and 26 it is concluded that for specified dead to live load ratios ($w = G/P$) and ultimate load factors (c) the compatibility condition reduces to a specific relationship between the steel percentages (p), the redistribution factors (r) and the yield safety parameters (x) of critical sections.

A typical p - r - x relationship is plotted in Fig. 9 for a rectangular R. C. section, reinforced in tension only with mild steel having a yield point of f_y = 40.000 psi, and assuming a value of 1/30 for the parameter $\beta\lambda$. In the left-hand side of the diagram the Eq. 26 is given graphically for various w/c ratios. In the right-hand half, Eq. 25c is plotted for various concrete grades (f'_c = 3,000 psi, 4,000 psi, and 5,000 psi).

The condition to have no plastic hinges at working-loads requires $c_{1i} > 1$, or $x_i > 1/c$ [1]. Since according to the above rule x is greater than or equal to 0.7, from Eq. 26 it is seen that the maximum redistribution factor implied is $r = 1.425$, for $x = 0.7$ and $w/c = 0$. Therefore, limit design solutions following the rule given and preventing the occurrence of plastic hinges at working loads range inside the rectangle bounded by $x = 0.7$ and $r = 1.425$ in Fig. 9. Now, for the solution to be compatible the redistribution factor must not exceed the value indicated by Eq. 25c. But, since r is imposed by the mentioned design rule, the condition of compatibility is turned into an upper limitation of the steel percentage of the critical section.

Thus, for a problem where $w/c = 0$, $x = 0.75$, and f'_c = 3,000 psi, the dotted line on Fig. 9 in the direction of the arrows shows a maximum permissible value of p of approximately 2.0%.

In brief the use of limit design for continuous R. C. beams involves generally two distinct steps: (1) Derivation of the plastic moments, $\overline{X}_i$, according to a convenient limit design theory (thus x_i and p values resulting); and (2) checking that under the given conditions the adopted steel percentages are below the upper limit indicated by Eqs. 25c and 26.

The first step may be performed by using the above mentioned rule. The second step requires p - r - x diagrams for current conditions in practice.

Three such diagrams were prepared under the specific assumptions indicated in Figs. 9, 10, and 11. These diagrams illustrate the influence of the following factors upon the compatibility condition:

(1) The concrete grade—positive effect of high grades (Fig. 9);

(2) the steel grade—negative effect of high grades (Fig. 10);

(3) the compression reinforcement—positive effect of strong compression reinforcement (Fig. 11).

The latter effect is of considerable practical significance, enabling a local adjustment of R. C. plastic hinges to suit the compatibility condition, but negligibly affecting the plastic moment distribution.

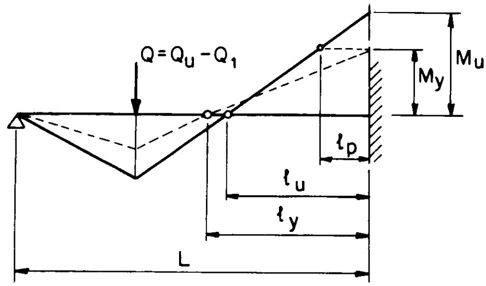


FIG. 8

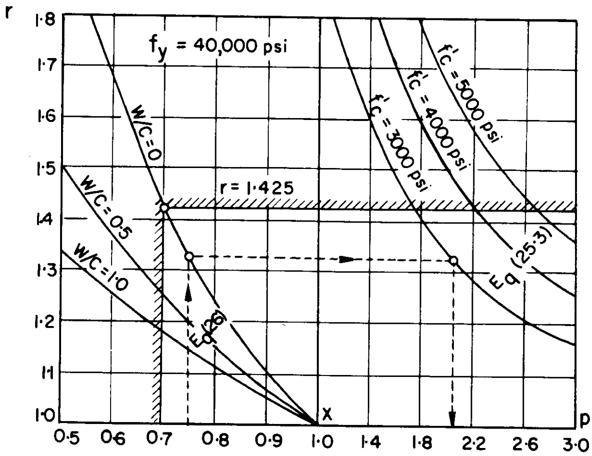


FIG. 9

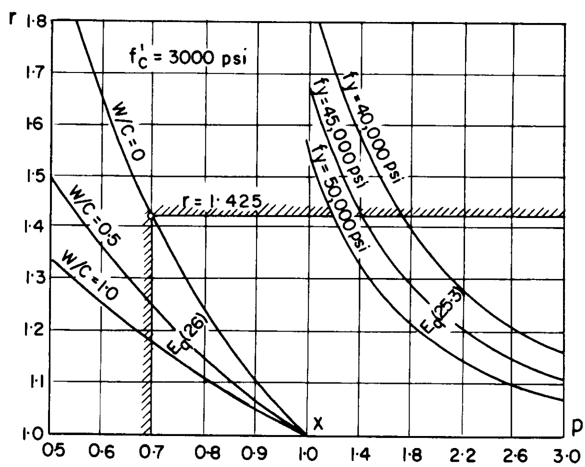


FIG. 10

It is interesting to note that for steel with f_y less than 40.000 psi, concrete grades larger than 3,000 psi and the worst redistribution conditions usually accepted (r = 1.425 for w/c = 0) the maximum steel percentage allowed is

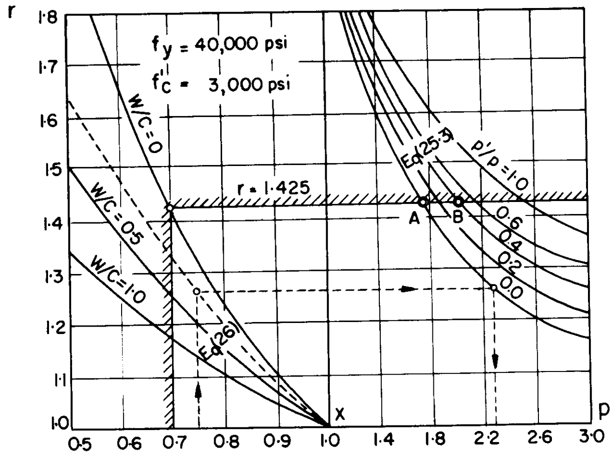


FIG. 11

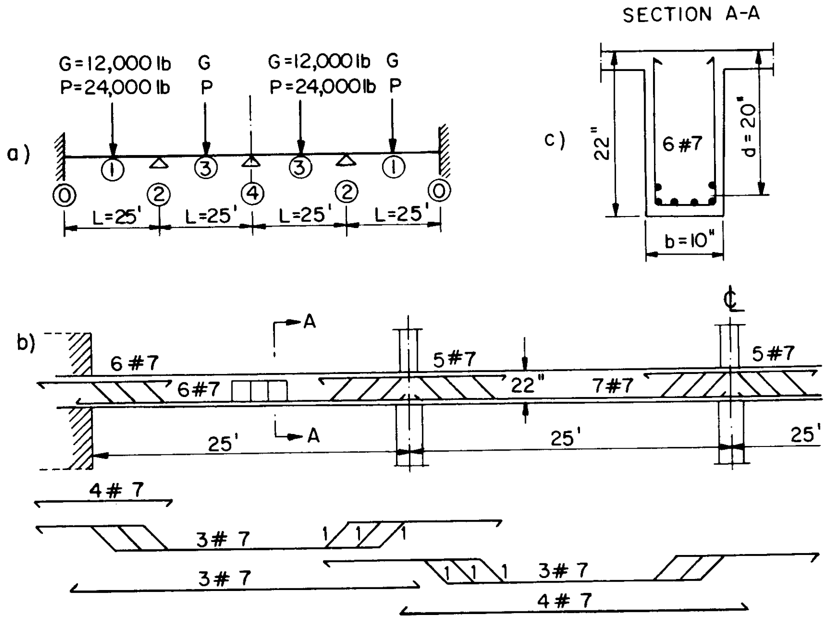


FIG. 12

of about p = 1.8%, while an addition of only p' = 0.4 p compression steel raises to p = 2% the permissible steel percentage (points A, B on Fig. 11).

This suggests that under the above usual conditions the compatibility problem is of no concern when critical sections of limit designed continuous beams are provided with steel percentages not exceeding about 2% in tension [1].

EXAMPLE

The limit design of a fixed-ended reinforced concrete beam, continuous over four spans of 25 ft each, is required (Fig. 12). The central point loading consists of a dead load G = 12.000 lb and a live load P = 24.000 lb. An ultimate load factor c = 2, f'_c = 3,000 psi, and f_y = 40.000 psi are specified.

Maximum elastic moments at critical sections are of the form $M_i = a_i\,GL + b_i\,PL$, the elastic constants a_i and b_i being as given in Table 1. According to the OLD rule indicated, the appropriate yield safety parameters will be x = 0.75 for support sections and x = 0.9 for span sections. The resulting plastic moments $\overline{X}_i = a_i\,GL + b_i\,x_i\,cPL$ are listed in column (6) of Table 1.

TABLE 1.—LIMIT DESIGN OF A R. C. CONTINUOUS BEAM[a]

Section	a_i	b_i	x_i	$b_i x_i c$	$10^{-6}\overline{X}_i$, in lb in.	$\dfrac{\overline{X}_i}{bd^2 f'_c}$	p_{nec}, in %	Effective Reinforcement			
								Tension		Compression	
								#	p%	#	p'/p
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0	0.125	0.170	0.75	0.255	2.29	0.191	1.70	6#7	1.80	3#7	6.5
1	0.125	0.147	0.90	0.264	2.37	0.198	1.72	6#7	1.80	-	-
2	0.125	0.147	0.75	0.220	2.04	0.170	1.45	5#7	1.50	3#7	0.6
3	0.125	0.174	0.90	0.313	2.71	0.226	2.00	7#7	2.10	-	-
4	0.125	0.156	0.75	0.234	2.14	0.178	1.50	5#7	1.50	4#7	0.8

b = 10 in. d = 20 in. G = 12.000 lb P = 24.000 lb
f'_c = 3,000 psi f_y = 40.000 psi c = 2 L = 25 ft

[a] Example, see Fig. 12.

Taking a R. C. section with b = 10 in. and d = 20 in., the reinforcement will be found using the ultimate moment values given by Eq. 21, neglecting initially the compression steel and assuming k_2/k_1k_3 = 0.59. With the values of $\overline{X}_i/bd^2\,f'c$ in column (7) of Table 1 the necessary steel percentages in tension are obtained (column 8).

The effective reinforcement of the beam is shown in Fig. 12b and in columns (9) to (12) of Table 1.

Now it is possible to check the compatibility condition. For the data of the problem w/c = 0.25 and from Fig. 11 it is seen (dotted line on the diagram) the maximum allowable p to be of about 2.3%. Since this steel percentage is exceeded at no plastic hinge, and compression steel is also provided at supports the solutions in Fig. 12 checks the compatibility requirement.

EVALUATION

Rotation Compatibility of Span Elastic Hinges.—From the previous arguments it is concluded that (1) loading scheme, (2) end supporting, (3) geometrical dimensions of steel and concrete, and (4) properties of materials, are the main factors affecting the compatibility problem of support critical sections.

Though just the same factors are to be considered for the span critical sections, available plastic rotations of the latter are considerably larger than of the former, mainly because the length of plastic zones are much more extended in spans than at support.

On the other hand span sections, because of the slab effect, have larger available plastic curvatures, while usually the plastic moments, according to OLD procedures [1], [6], [7], [28] differ but negligibly of the elastic moments.

For these reasons the rotation compatibility may be supposed as warranted for span critical sections of usual continuous R. C. beams and therefore it requires no special consideration.

The following are factors affecting the rotation capacity of plastic hinges:

(a) The shape factor (β) (or curvature distribution factor) depends on the loading scheme of the beam, the variation of the actual flexural rigidity along the member, the longitudinal and transversal reinforcement, etc.

Actual values of β can be derived from tests only, the range of validity of results being, however, limited to the specific conditions of the experiments.

Suggested values for β according to Chan are 0.35 and 0.2 for unbound and bound sections, respectively. However these figures seem to be excessively low.

In a series of recent tests on continuous two-span beams values of 0.3 to 0.8 were currently obtained. It appears therefrom that in the absence of axial forces β = 1/2 is a reasonable value for practical purposes.

(b) The length of plastic hinges (l_p) is difficult to evaluate in most general terms, because of its sensitivity to the loading scheme, and other factors,

When concentrated loads are applied on the beam l_p takes, however the simple form (refer to Fig. 8)

$$\frac{l_p}{l_u} = 1 - \frac{M_y}{M_u} \quad \text{.................} \quad (27)$$

in which l_u = the distance from the support plastic hinge to the point of contraflexure at ultimate load. This is known when the loading scheme and the ultimate moments M_u for support and span critical sections are known.

Despite the major effect of l_p value on the rotation capacity of R. C. plastic hinges, there are still too little available data for its proper evaluation taking account of all the variables affecting it.

The few tests reported on this subject, notably those of Ernst [23], Chan [13] and Yamada [24], show that a large amount of experimental work should be carried out in order that reliable and safe data be provided.

So far various semi-empirical proposals giving the value of l_p illustrate a considerable diversity of views. Thus initially Baker proposed a safe-limiting value of $l_p = d$[2], Chan recommended $l_p/l_u = 0.4$ [13], and subsequently $l_p/l_u = 0.25 \ldots\ldots 0.35$ [22], Yamada [24] suggested $l_p/d = 2(1-k_2)$, the Soviet provisions for the design of R. C. structures allowing for stress redistribution [29] recommended $l_p = l_c$, which l_c = the distance between the two successive cracks, as given by Mashov's theory [14], finally the ICE Research Committee [21] suggested a new relation $l_p/d = k_1k_2k_3\,(l_u/d)^{0.25}$ which is advocated also in a recent paper by Chan [29].

In the present paper, a mean value of $l_p = 1/15$ L has been adopted in the calculations. Therefore the only arbitrary parameter introduced is the product $\beta\lambda = 1/30$, a value which seems to be quite conservative.

(c) The plastic curvature (ϕ_p) depends on the deformational properties of the concrete section and the depth of the compressed zone at yield and ultimate loads (Eq. 12). ϕ_p is therefore affected by the factors governing the values of k_y and k_u, i. e., the shape of concrete section, grades of steel and concrete, amount of tension, and compression reinforcement, etc.

Consideration of Eq. 13a makes it clear that since ϵ_{cu} has practically a constant value, in order to increase θ_p it is desirable to have as smaller

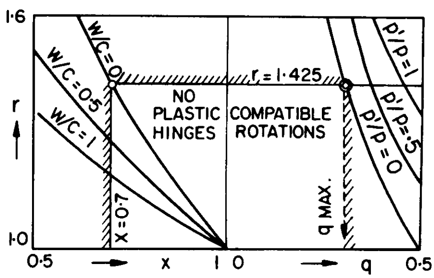


FIG. 13

values of k_u as possible. This may be achieved by using: (a) High grades of concrete; (b) lower yield point stresses for steel; and (c) compression reinforcement.

The practical significance of the latter measure will be obvious noting that in all cases a certain amount of compression steel must be provided at supports, increasing thus the plastic curvature and the rotation capacity of support plastic hinges.

Development of a Simpler Solution.—A valuable simplification of the compatibility check would result if a single diagram might be used instead of the set of diagrams in Figs. 9, 10, and 11.

This would easily be possible if rotation parameters $\phi_u/\phi_y - 1$ in Eq. 25b had values of the type suggested by Wright and Berwanger [17]. Indeed, if according to reference [17] $\phi_u/\phi_y = 1/\rho q_y^2$ or $1/\rho'\, q_y q_y'$ for sections without or with compression reinforcement, the compatibility condition (Eq. 25b) would become $r \leq 1 + 3\,\beta\lambda\left(1/\rho\, q_y^2 - 1\right)$ or $r \leq 1 + 3\,\beta\lambda\left(1/\rho'\, q_y\, q_y' - 1\right)$ for the two cases, respectively.

Under these circumstances a unique q - r - x diagram would be obtained, having the typical aspect of Fig. 13.

It is seen that a single diagram would enable all the variable parameters (w/c, x, r, p, p', fy, f'c) to be allowed for simultaneously, provided the constants ρ and ρ' to be accurately assessed, either from theory or from tests.

It is interesting to note that under the most severe redistribution conditions (i.e., for w/c = 0, p'/p = 0, and x = 0.7) compatible solutions of limit design range inside the dashed rectangle in Fig. 13. The entire problem is thus reduced to an upper limitation of the reinforcement parameter $q = p\, f_y/f'_c$. The same conclusion is provided in the Soviet specifications [30], though on quite different considerations than those suggested above.

SUMMARY AND CONCLUSIONS

Under reasonable, safe assumptions the condition of rotation compatibility may be reduced to a limitation of the redistribution factors (or of the yield safety parameters) with regard to each plastic hinge of a continuous beam.

Since in "optimum limit design" a definite value for the yield safety parameter is specified (in order to achieve the best service conditions) the compatibility requirement results in an upper limitation of the tension steel reinforcement.

As a corollary the "optimum limit design" of continuous beams may be performed disregarding the compatibility problem, as shown in references [1], [6], and [7]. The only thing that is required is to check, in the final stage of the design, that the adopted steel percentages at critical sections do not exceed the upper limits given by Eq. 25c.

Moreover, the compatibility of span critical sections being warranted, due to their considerable deformational capacity, only the support critical sections must be subject of the compatibility check.

This is easily performed when the p - r - x diagrams are available, relating the tension steel percentages, the redistribution factors and the yield safety parameters to the grades of concrete and steel, the ratio of compression to tension steel percentages and the ratio of dead to ultimate live load ratio. Figs. 9 to 11 are typical examples of such diagrams.

However a generalized-unique type of diagram (as in Fig. 13) would become possible as further studies will enable a more accurate evaluation of the constants involved in the rotation capacity of reinforced concrete plastic hinges.

The condition of compatibility presented in the paper and the p - r - x diagrams summarizing the theory involved outline the favorable effects on the moment redistribution of the compression steel, of a lower grade of steel and of a higher grade of concrete.

It is believed that it will be possible to extend the technique described in the paper for checking the compatibility to arbitrary types of redundant beams. Though the amount of errors introduced by the adopted assumptions will vary according to the particular cases considered, the basic idea consisting in the reduction of the compatibility problem to a limitation of the steel percentages at critical sections will still retain its full practical interest.

With regard to the errors involved in the compatibility check (and particularly in the calculations of θ_p) it is to be noted that even if they were not subject to inevitable approximations, the actual properties of the concrete used in construction remain essentially unknown, irrespective of the properties assumed or effectively measured on cube or cylinder specimens.

In the writer's opinion, a too involved deformational analysis cannot bring too much accuracy because of the inherent imperfections in the present knowledge on the sense and the amount of error introduced in computations both by the assumptions accepted and the actual properties of building materials.

It is for these reasons that simple means, as those described, to estimate rotation compatibility appear as more desirable than too elaborate analyses, provided, of course, that the qualitative view of the phenomena is accurately reflected and the numerical solution for a given problem remains within reasonable limits of safety and economy.

APPENDIX I.—NOTATION

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b = width of a rectangular R. C. section;

c = ultimate load factor;

c_{1i} = yield load factor with respect to critical section i;

c_i, c_j = elastic constants reflecting the influence of end stiffenings of a member on the slopes at i and j, respectively.

d, d' = depth to the most stressed concrete fiber of the tension and compression steel, respectively;

E_s, E_c = young moduli for steel and concrete, respectively;

f_c = effective stress in the extreme concrete fiber;

f'_c = cylinder strength;

f_s, f'_s = effective stress in tension and compression reinforcement, respectively;

f_y = yield point stress of steel;

G = dead load;

i, j, k = indices referring to the order of critical sections of a beam;

k_1, k_2, k_3 = constants defining the stress block;

L = span length;

l_p = length of a plastic hinge;

l'_p = equivalent length of a plastic hinge;

l_y = distance between support and point of contraflexure at yield of the support;

l_u = the same, when the ultimate moment is reached at support;

M_o = ordinate of the force bending moment diagram;

M_i = elastic moment at section i assuming constant rigidity along the beam;

M'_i = the same, but taking into account the end stiffening;

M_y, M_u = yield and ultimate moments of a critical section;

m_i = loading parameter defining the fixed-end moments for a beam with constant rigidity;

$n = E_s/E_c$ = modular ratio;

P = live load;

$p = A_s/bd$, $p' = A'_s/bd$ = tension steel ratio; compression steel ratio;

$Q_{li} = G+c_{li}P$ = total yield load;

$Q_u = G+cP$ = ultimate load;

$q = pf_y/f'_c$, $q' = p'f_y/f'_c$ = tension reinforcement index; compression reinforcement index;

$r_i = \frac{G+P_u}{G+P_{li}}$ = redistribution factor with respect to the critical section i;

r'_i = the same, but for the beam with stiffened ends;

s_i = loading constant reflecting the influence of end stiffenings of a member on the slopes at i due to the application of the load;

u = index referring to the ultimate load stage of the beam;

$w = G/P$ = ratio of dead to live load;

$\overline{X}_i$ = plastic moment of section i;

x_i = yield safety parameter = c_{li}/c (ratio of the yield load factor of section i to the ultimate load factor of the structure);

y = index referring to the yield of considered section;

z = distance from the support section to a current section of the beam;

$\delta' = d'/d$

ϵ_{cy}, ϵ_{cu} = concrete strain at yield of tension steel and crushing of concrete, respectively;

$\epsilon_p = \epsilon_{cu} - \epsilon_{cy}$ = plastic strain of concrete;

$\epsilon_{sy} = f_y/E_s$ = yield strain of steel;

$\phi = M/R = \epsilon/kd$ = curvature of the neutral axis in a R. C. section;

ϕ_y, ϕ_u = curvature of support section at its yield and ultimate load, respectively;

$\phi_p = \phi_u - \phi_y$ = plastic curvature of support plastic hinges;

$\lambda = l_p/L$ = relative length of a plastic hinge;

$\mu = p'/p$ = ratio of compression and tension reinforcements;

θ_i = inelastic rotation of the critical section i assuming constant rigidity of the beam;

θ'_i = the same, but allowing for the stiffening of the ends;

θ_{pi} = rotation capacity of plastic hinge i.

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