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Rectangular Concrete Stress Distribution in Ultimate Strength Design

By ALAN H. MATTOCK, LADISLAV B. KRIZ,
and EIVIND HOGNESTAD

An ultimate strength design theory of broad applicability is developed, based on an equivalent rectangular stress distribution in the concrete compression zone and in general accord with the Appendix to the 1956 ACI Building Code. The theory is characterized by simplicity without significant loss of accuracy.

The proposed method of ultimate strength design is applied to a wide variety of structural concrete beams and columns, subject to various combinations of bending and axial load. Calculated ultimate strengths are compared with experimentally determined ultimate strengths for a wide range of variables, and an excellent agreement results.

It is concluded that the proposed extension of the rectangular stress distribution theory permits prediction with sufficient accuracy of the ultimate strength in bending and compression of all types of structural concrete sections likely to be encountered in structural design practice, including odd-shaped sections and other unusual cases.

Part 1 — Review of Basic Assumptions

■ IN THE OCTOBER, 1955, REPORT OF ACI-ASCE Committee 327 on ultimate strength design as abstracted in the Appendix to the 1956 ACI Building Code (ACI 318-56), ultimate strength design methods were given in specific terms only for the cases most frequently met in design practice. Specific design methods were not given for odd-shaped cross sections and other special cases. Secondly, extensive researches in the United States and abroad have been completed since the Committee 327 report was prepared. A re-evaluation of the design principles involved is therefore desirable. Finally, the rectangular stress distribution theory has become widely used in design practice and in general accord with the 1956 Code. The extent to which this simple theory can be safely extended to unusual design cases has previously not been thoroughly studied.

At the request of Subcommittee 15 on ultimate strength design of ACI Committee 318, Standard Building Code, and in the light of recent research findings, the authors have studied the rectangular stress distribution theory as commonly used in design practice under the 1956 Code. In addition, they have extended this theory into a *generally applicable* and reasonably simple design tool based on a single set of assumptions.

This paper presents such assumptions and substantiating evidence for consideration by ACI Committee 318 and the ACI membership as a contribution toward development of a future revision of the ACI Building Code. In this manner, the recommendations and views expressed are those of the authors and do not necessarily reflect the collective judgment of Committee 318.

ASSUMPTIONS IN ULTIMATE STRENGTH DESIGN

By use of the following general assumptions, ultimate strength of sections subjected to combined bending and axial load can be predicted with adequate accuracy, even for odd-shaped cross sections and other unusual cases:

1. At ultimate strength, a concrete stress of intensity 0.85 times the concrete cylinder strength may be assumed uniformly distributed over an equivalent compression zone bounded by the edges of the cross section and a straight line located parallel to the neutral axis at a distance k_1c from the region of maximum compressive strain. The distance c from the region of maximum strain to the neutral axis is measured in a direction perpendicular to that axis. The fraction k_1 is taken as 0.85 for concrete cylinder strengths up to 4000 psi and is reduced continuously at a rate of 0.05 for each 1000 psi of strength in excess of 4000 psi.
2. Tensile strength of the concrete may be neglected in flexural calculations.
3. Strain in the concrete at the various section levels may be assumed directly proportional to the distance from the neutral axis. Except in anchorage regions, strain in reinforcing bars may be assumed equal to the tensile or compressive strain in the concrete at the same distance from the neutral axis as the centroid of each bar or group of bars considered.
4. The maximum strain at an extreme edge of the concrete compression zone may be assumed equal to 0.003.
5. Stress in reinforcing bars below the yield point stress for the grade of steel used may be taken as 30,000,000 psi times the steel strain. For strain greater than that corresponding to the yield point stress, the reinforcement stress may be considered independent of strain and equal to the yield point stress.

USE OF SIMPLIFIED EQUATIONS

The equations for ultimate strength design given in the Appendix to the 1956 ACI Building Code, ACI 318-56, can almost all be derived from the assumptions set out above. Certain of the Code equations for the design of columns contain additional simplifying assumptions.

Alan H. Mattock (ACI member) joined the staff of the Portland Cement Association Research and Development Laboratories, Skokie, Ill., in 1957. After graduation from London University, he served with the Fire Research Station at Elstree, England, and in the Lancashire County Bridgmaster's design office. He returned to study and was awarded MS (Eng), London University, in 1949. For the next 3 years he was a district engineer in British Guiana, South America, where the principal work was the construction of sea and river walls, and drainage and irrigation works.

From 1952 to 1957 Dr. Mattock was a lecturer at Imperial College, University of London, having charge of the Concrete Technology Laboratory; while there he carried out research and consulting work, and in 1955 was awarded PhD, London University. Dr. Mattock is a member of ACI-ASCE Committee 323, Prestressed Concrete, and also of ACI-ASCE Committee 328, Limit Design.

Ladislav B. Kriz (ACI member) graduated in civil engineering from the University of Illinois in 1957, with previous schooling at the Technical University in Prague, Czechoslovakia, and at the School of Civil Engineering in Madrid, Spain. His undergraduate studies were interrupted by brief periods of employment as draftsman with Zublin-Perriere, Paris, France; as a designer with Byrnes Associates, New York, N. Y.; and a 4-year tour of duty as surveyor in the U. S. Air Force.

He joined the staff of the Structural Section of the Portland Cement Association Research and Development Laboratories in 1957. During his employment with the association, he earned his MS degree from Northwestern University.

Eivind Hognestad (ACI member), manager, Structural Development Section, Portland Cement Association Research and Development Laboratories, is well known to ACI JOURNAL readers. Prior to joining the PCA staff in 1953, Dr. Hognestad was research associate professor of theoretical and applied mechanics at the University of Illinois.

He obtained his MS degree in civil engineering from the University of Illinois and the degrees of civil engineer and doctor of technical sciences from the Norwegian Institute of Technology.

Dr. Hognestad is chairman of ACI-ASCE Committee 326, Shear and Diagonal Tension; a member of ACI Committee 318, Standard Building Code, and ACI Committee 336, Combined Footings.

For the straight-forward cases, for which these Code equations have been derived, the equations are entirely adequate and may be used with confidence. It should be noted however, in using Eq. (A3) from the Appendix to the 1956 ACI Code for the design of a beam with compression reinforcement, that the stress in the compression steel at ultimate strength should be checked to confirm that yielding of the steel has in fact occurred.

In unusual problems of design, the basic assumptions given may be used directly in everyday practice. Cases not susceptible to direct mathematical solution may be treated by suitable iteration procedures. In mathematically complex cases, calculations based on the basic assumptions may be used together with experimental data to develop and verify simplified procedures for design office use, such as those proposed in the writings of P. M. Ferguson and C. S. Whitney.

Notation

Wherever possible the notation used in the ultimate strength design appendix to the ACI Building Code (ACI 318-56) has been used in this paper. For convenience, the notation is summarized as follows:

A_c = net concrete cross section, or area of segment of circle covered by equivalent concrete stress distribution
A_s = area of tensile reinforcement
A_s' = area of compressive reinforcement
A_{sf} = steel area required to develop compressive strength of overhanging flange in T-sections
A_{st} = total area of longitudinal reinforcement
a = depth of equivalent rectangular stress distribution
b = width of a rectangular section, or over-all width of flange in T-sections
b' = width of web in T-sections
C = resultant concrete compressive force
c = distance from extreme compressive fiber to neutral axis at ultimate strength
D = diameter of a circular section
d = distance from extreme compressive fiber to centroid of tensile reinforcement
d' = distance from extreme compressive fiber to centroid of compressive reinforcement
E_s = modulus of elasticity of reinforcing steel
e = eccentricity of axial load, measured from the centroid of tensile reinforcement, unless otherwise specified
F = ratio of change in strain of steel of prestressed beam to change in strain of concrete at the level of the steel
F_{sc} = resultant steel compressive force
F_{st} = resultant steel tensile force
f_c' = 6 x 12-in. concrete cylinder strength
f_s = stress in tensile reinforcement at ultimate strength
f_s' = stress in compressive reinforcement at ultimate strength
f_y = yield point stress of tensile reinforcement
f_y' = yield point stress of compressive reinforcement
G_c = center of action of concrete compressive force
G_{sc} = center of action of steel compressive force
G_{st} = center of action of steel tensile force
k_u = c/d
k_1 = ratio of average stress to maximum stress
k_2 = ratio of depth to resultant of concrete compressive force, to depth of neutral axis
k_3 = ratio of maximum stress to 6 x 12-in cylinder strength, f_c'
M_{ult} = ultimate moment of resistance
m = plastic modular ratio, $\varepsilon_u E_s / 0.85\, k_1 f_c'$
P_u = ultimate strength of eccentrically loaded member
P_o = ultimate strength of concentrically loaded member
p = tensile steel ratio, A_s/bd
p' = compressive steel ratio, A_s'/bd
p_b = steel ratio at balanced ultimate strength condition in a beam without compression reinforcement
p_B = steel ratio at balanced ultimate strength condition in beam with both tension and compression reinforcement
p_f = $A_{sf}/b'd$
p_w = tension steel ratio for T-beams, $A_s/b'd$
p_{wb} = tension steel ratio for T-beam with balanced ultimate strength condition
q = tension reinforcement index, pf_y/f_c'
q' = compression reinforcement index, $p'f_y'/f_c'$
q_b = tension reinforcement index for balanced ultimate strength conditions, $p_b f_y/f_c'$
q_w = tension reinforcement index for T-beams, $p_w f_y/f_c'$
q_{wb} = tension reinforcement index for T-beams with balanced ultimate strength condition, $p_{wb} f_y/f_c'$
t = flange thickness in T-beams, also total depth of rectangular section column

α = inclination of neutral axis to horizontal in nonsymmetrical section

ε = strain

ε_c = concrete tensile strain at ultimate at level of steel in prestressed beam

ε_{cp} = concrete precompression strain at level of steel in a prestressed beam

ε_u = maximum concrete compression strain at ultimate beam strength

ε_s = tensile steel strain at ultimate beam strength

ε_y = tensile steel yield strain

ε_s' = compression steel strain at ultimate strength

ε_y' = compression steel yield strain

ε_{se} = effective steel prestrain

ε_{sn} = strain in steel distance, a_n, from neutral axis

θ = half included angle between two upper faces of beams with triangular compression zones

CONCRETE STRESS DISTRIBUTION

The general form of the concrete compression stress distribution at ultimate strength in a reinforced concrete member is shown in Fig. 1. The properties of the "stress block" are represented by the following coefficients:

k_1 = ratio of average stress to maximum stress

k_2 = ratio of depth to resultant of compressive force, to depth to neutral axis

k_3 = ratio of maximum stress to 6 x 12-in. cylinder strength, f_c'

Historical background

The use of a design theory based on the ultimate strength of sections is in effect a return to the original concept of design, in that early design formulas were empirical, being based on the failure loads of typical elements as found by experiment.

The first published ultimate load theory was that of Koenen[1] who in 1886 assumed a straight line distribution of concrete stress and a neutral axis at middepth. Since that time about 30 theories have been published. The salient points of many of these theories were set out in *Bulletin* No. 399,[8] University of Illinois Engineering Experiment Station.

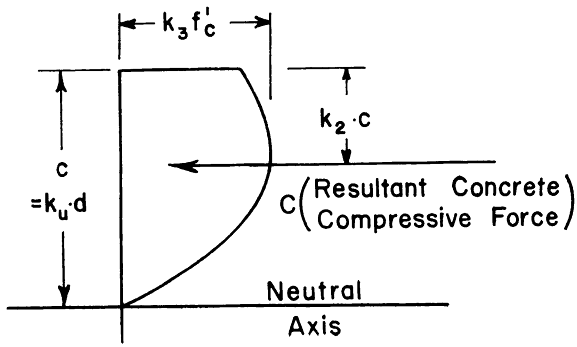


Fig. 1—Concrete stress distribution at ultimate strength

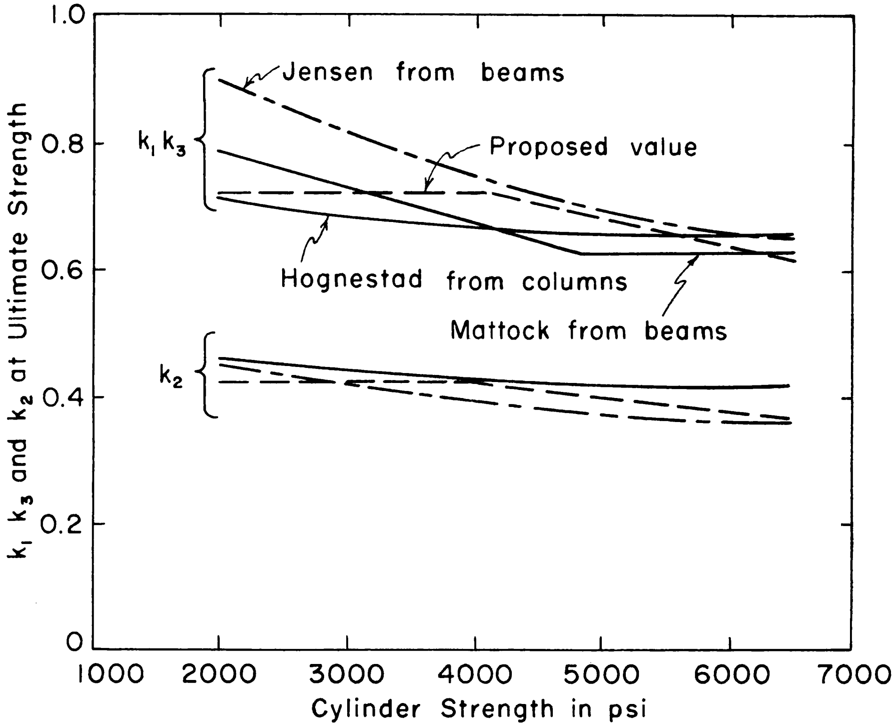


Fig. 2—Design values from tests of reinforced concrete

Many different distributions of stress in the concrete compression zone have been suggested, but in the light of recent experimental[2] and analytical[3] investigations, it is considered that the use of the "equivalent rectangular stress block" in design calculations will yield sufficiently accurate results and will at the same time lead to considerable simplification of design calculations. The use of a rectangular concrete compression stress block was first proposed by von Emperger[4] in 1904, and since that time by several other engineers, the best known in this country being C. S. Whitney, whose paper[5] in 1937 was a notable contribution to the literature of ultimate strength design.

Reinforced concrete investigations

Until recently, most of the available information regarding stress distribution in concrete was derived from tests of reinforced concrete members. In the early 1930's the extensive ACI investigation[6] of concentrically loaded columns led to the addition law which states that the ultimate strength of a column is equal to 85 percent of the cylinder strength times the concrete area plus the yield stress of the longitudinal steel times its area. Thus, for concentrically loaded columns, the value of $k_3 = 0.85$ was derived experimentally.

Several papers on stress distribution and ultimate strength design were published in Europe during the 1930's, and these were followed by the studies of ultimate strength in this country by Whitney.[5] His analytical approach involved an equivalent rectangular stress block with a maximum stress of 0.85 f_c', thus his value of k_3 was 0.85.

In 1943 an extensive study of the ultimate strength of beams was presented by Jensen.[7] He proposed a trapezoidal idealized stress distribution, and derived the properties of this trapezoid as a function of cylinder strength by analysis of the observed ultimate strength of reinforced concrete beams. On this basis he found the values of k_1k_3 and k_2 shown in Fig. 2.

A study of the ultimate strength of eccentrically loaded columns was reported by Hognestad[8] in 1951. The stress distribution used consists of a rising parabola and a descending straight line after the maximum stress. From the results of concentrically loaded columns, a maximum stress equal to 0.85 f_c' was chosen, that is, $k_3 = 0.85$. The slope of the descending straight line was chosen so as to give the best statistical agreement between calculated and observed column strengths in 120 tests. The corresponding values of k_1k_3 and k_2 are also shown in Fig. 2.

Additional experimental evidence as to the parameters of the concrete stress block was presented in 1956.[9] Using the measured values of depth to the neutral axis at failure and of the ultimate moment of resistance, and assuming a safe limiting value 0.43 for k_2, values of k_1k_3 were calculated for 69 beams tested at Imperial College, University of London, and at the University of Illinois. Based on these results the values of k_1k_3 labeled "Mattock" in Fig. 2 were proposed as suitable for ultimate strength design.

Plain concrete investigations

In recent years various tests have been carried out on plain concrete specimens, using special testing techniques, in an effort to obtain a true picture of the stress-strain relationship for concrete during loading to failure. Independently, but almost simultaneously, tests of eccentrically loaded prismatic concrete specimens were carried out in the Portland Cement Association laboratories[10] and by Rüsch[11] at the Munich Institute of Technology.

The specimen used in the PCA tests is shown in Fig. 3. The two thrusts P_1 and P_2 were varied independently, in such a manner that the neutral axis was maintained at the bottom face of the specimen throughout the test. By equating the internal and external forces and moments, it was possible to calculate the values of k_1k_3 and k_2 directly.

In the Munich tests, groups of about five identical prisms were tested with a different and constant eccentricity for each test. By plotting, for all specimens within one group, strain measured at an outside

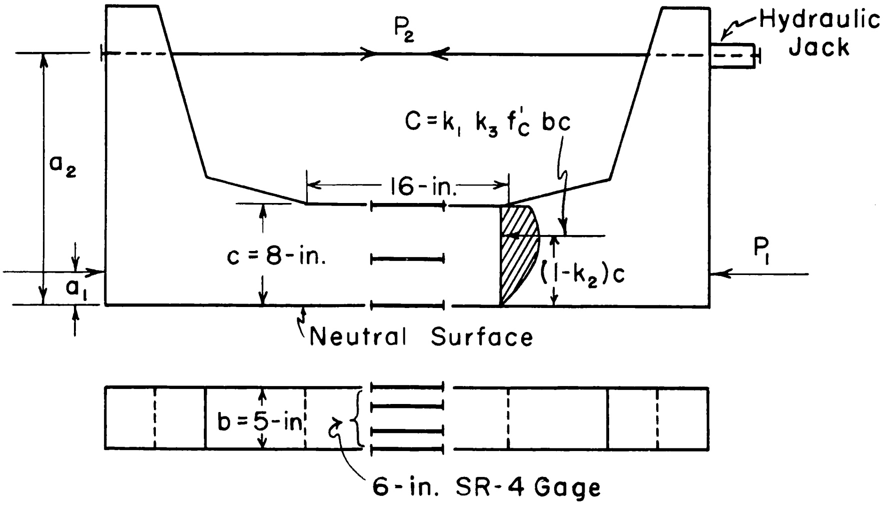


Fig. 3—PCA eccentric load specimen

face at ultimate strength versus applied eccentricity, the eccentricity corresponding to a position of the neutral axis at an edge of the section was determined. By applying statistical methods, the magnitude and position of the internal concrete force at ultimate strength was determined.

The parameters of the concrete stress block obtained at the two laboratories are summarized and compared in Fig. 4 with the values proposed in this paper. The PCA tests with sand-gravel aggregates have been reported in detail elsewhere,[10] a detailed report on the tests by J. A. Hanson with lightweight aggregates has not yet been published. The German tests[11] were made with sand-gravel aggregates. It can be seen that test results for sand-gravel concretes are in good agreement, and that the radical change in aggregate type to lightweight materials caused only a minor change in the stress distribution properties.

Analytical investigations

In a recent analytical paper[3] the problem of ultimate flexural strength of reinforced concrete members was reduced to finding the maximum value of a load function expressed in terms of the internal resisting forces of the loaded member. Assuming only that concrete stress, f, is some function of strain, ε, given by $f = F(\varepsilon)$, and that plane sections remain plane during bending, the moment of resistance of a reinforced concrete section was expressed in terms of the extreme edge concrete strain, ε_c, the reinforcement yield point, f_y, and the dimensional properties of the cross section. This expression for moment was differentiated with respect to the extreme edge concrete strain and equated to zero. The resulting equation demonstrates that, when the maximum moment in a rectangular reinforced concrete beam failing in tension under symmetrical bending is reached, then the concrete stress distribution is such that the total compressive force is equal to that obtained from an equivalent rectangular stress block in which the stress is equal to the actual stress in the extreme compression fibers, and which has a depth equal to twice the distance from the extreme compressive fibers to the center of action of the resultant concrete compressive force. From a study of concrete stress-strain curves obtained in the PCA eccentrically loaded prism investigation,[10] it was deduced that the relationship between concrete stress at extreme compressive fiber, f_u, and the cylinder strength, f_c', can be expressed closely as

$$f_u = 5 f_c'^{0.8}$$

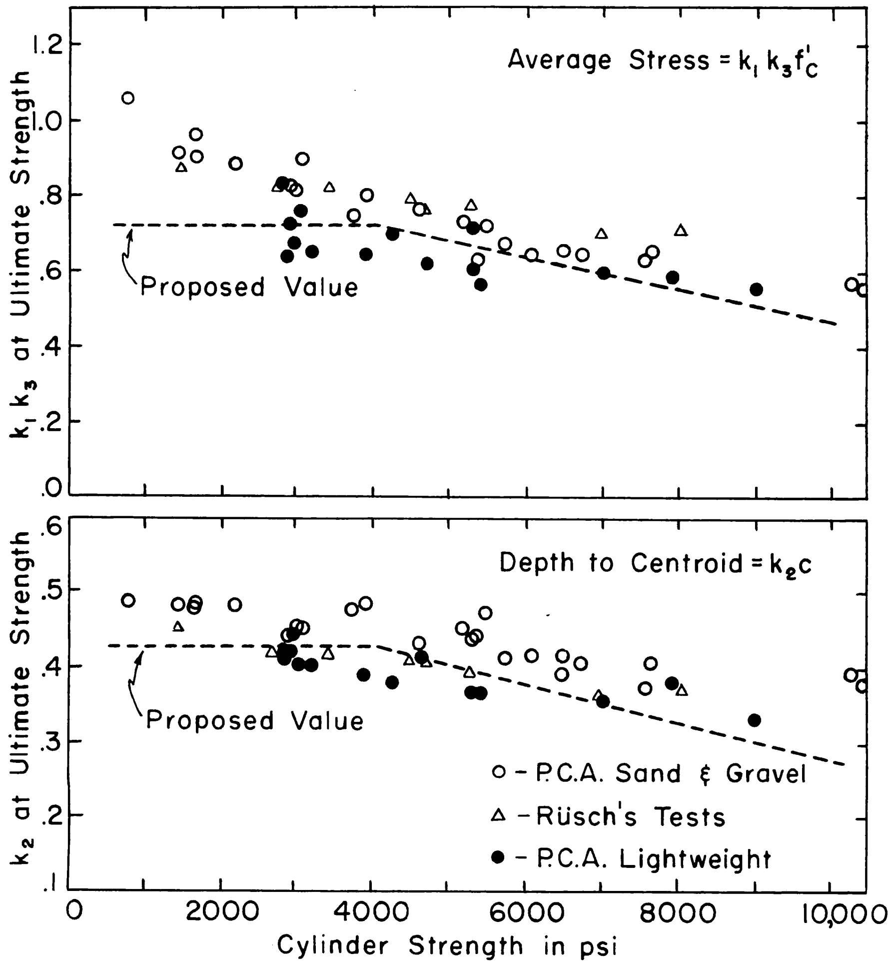


Fig. 4—Properties of concrete stress distribution at ultimate strength determined from tests on plain concrete

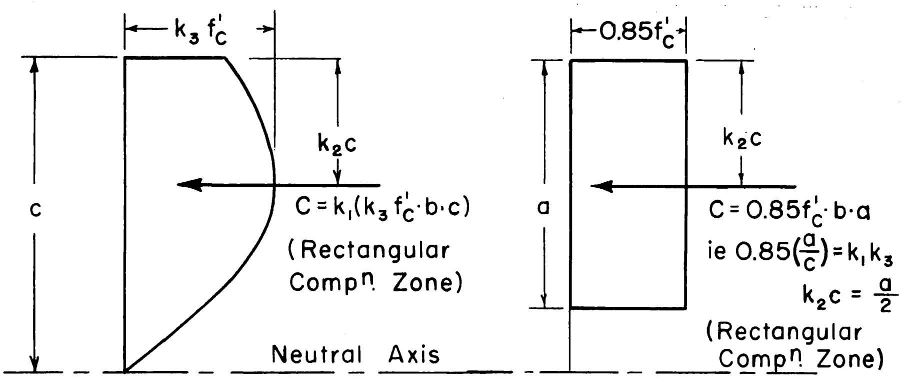


Fig. 5a (left)—Actual concrete stress distribution. Fig. 5b (right)—Equivalent rectangular stress distribution

However, for purposes of practical application a conservative straight-line relationship, $f_u = 0.85f_c'$ was proposed.

EQUIVALENT RECTANGULAR STRESS DISTRIBUTION

Nature of approximation

The essentials of the approximation are shown in Fig. 5 and may be summarized as follows: At ultimate strength, the compressive stress in the concrete compression zone is, for purposes of computation, assumed to be uniformly distributed from the region of maximum compressive strain to a depth, a, measured in a direction perpendicular to the neutral axis, where the depth, a, is less than the depth to the neutral axis, c. The uniformly distributed stress is taken to be equal to 0.85 of the cylinder strength, f_c'. The ratio, a/c is taken to be equal to 0.85 for concrete cylinder strength up to 4000 psi, and thereafter is reduced by 0.05 for each 1000 psi of strength in excess of 4000 psi.

For a rectangular concrete compression zone the average concrete compressive stress $k_1k_3f_c'$ is equal to $0.85(a/c)f_c'$, that is $k_1k_3 = 0.85(a/c)$, also $k_2 = ½(a/c)$. Since k_3 is taken as 0.85 f_c', it follows that $a/c = k_1$ and $a = k_1c$. For the values of a/c proposed above, k_1k_3 and k_2 will have values as indicated in Fig. 2 and 4. It is seen that the proposed values for k_1k_3, and k_2 correspond closely with the values determined by experiment.

Assumptions in ultimate strength design

The assumptions necessary in ultimate strength design were set out earlier in this paper, and will now be discussed item by item.

The nature of the first assumption has already been discussed above.

The second assumption is very nearly correct; any tension zone which does exist in the concrete at ultimate strength, in a normally proportioned reinforced concrete section, is very small indeed; and its lever arm is also small. It is therefore reasonable to neglect any contribution by concrete tension to the ultimate moment of resistance of a section.

The third assumption is not strictly correct for a reinforced concrete section after cracking, since the strain in the concrete on the tension side of the neutral axis will vary considerably, at any given level, due to cracking. If, however, we measure the extension per unit length of

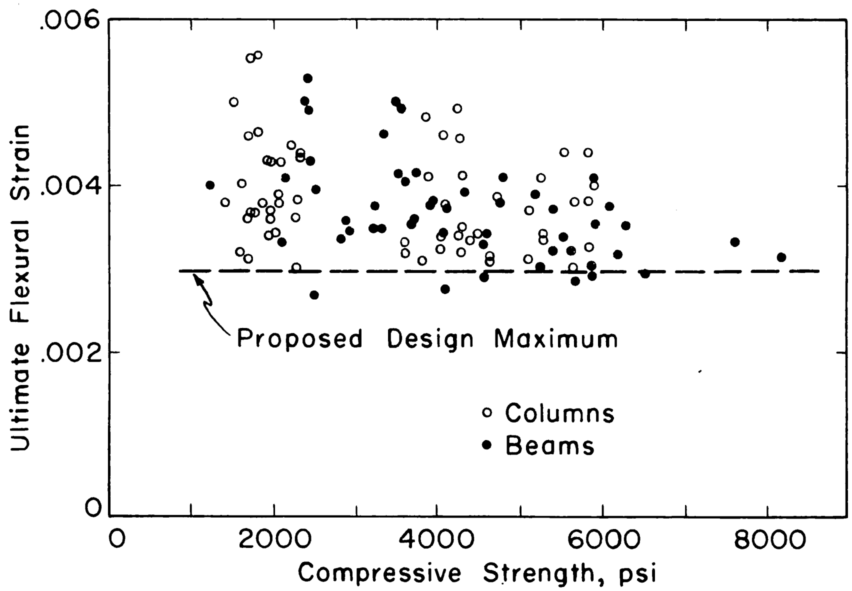


Fig. 6a—Ultimate strain from tests of reinforced members

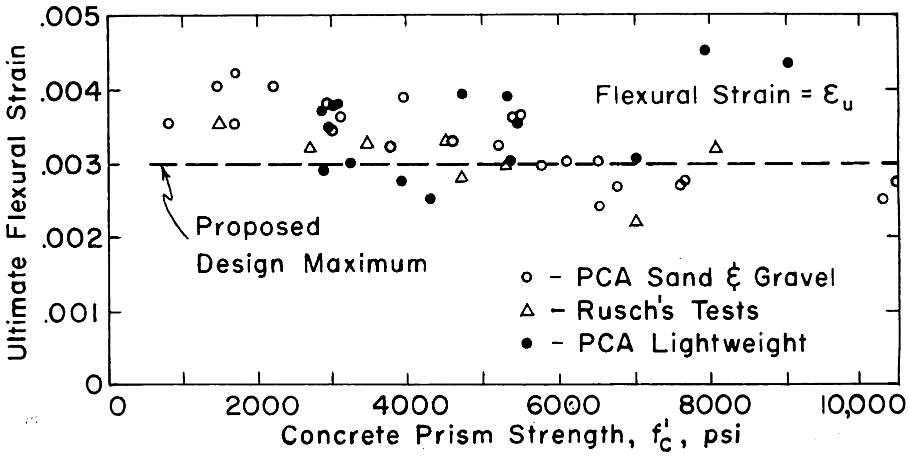


Fig. 6b—Ultimate strain from tests of plain concrete specimens

a gage length including several cracks, we find that this "apparent" tensile strain varies linearly with distance from the neutral axis. If for a particular section, we measure the compression strains and apparent tensile strains using the same gage length, then the distribution of these strains will be very close to linear. By experiment it has also been found that for the normal type of deformed bar reinforcement, the strain in the steel is nearly equal to the strain in the adjacent con-

TABLE 1—SUMMARY OF COMPARISONS OF CALCULATED AND EXPERIMENTAL ULTIMATE STRENGTHS

Type of member	Loading type and number tested	Failure mode	Average value Test ultimate strength / Calc. ultimate strength	Standard deviation
Reinforced concrete rectangular beams reinforced in tension	Simple bending, 59 beams	Controlled by crushing of concrete	All beams: 1.06 Beams with $f_c' \geq$ 2000 psi: 1.02	0.113 0.083
Reinforced concrete rectangular beams reinforced in tension and compression	Simple bending, 33 beams Simple bending, 11 beams	Controlled by yield of tension reinforcement Controlled by crushing of concrete and yield of compression steel	Beams using structural or intermediate grade steel: 1.04 Beams using high strength alloy steel: 1.21 All beams: 1.07	0.042 0.170 0.072
Prestressed beams with well bonded tendons	Simple bending, 32 beams	Varied	All beams: 1.03	0.077
Prestressed beams with unbonded tendons	Simple bending, 24 beams	Varied	All beams: 1.05	0.094
Symmetrical reinforced concrete T-beams reinforced in tension	Simple bending, 15 beams	Controlled by yield of the reinforcement	All beams: 1.11	0.100
Unsymmetrical reinforced concrete T-beams reinforced in tension	Simple bending, 12 beams	Varied	All beams: 1.10 Beams controlled by yield of steel: 1.04	0.138 0.047
Reinforced concrete beams with a triangular compression zone reinforced in tension	Simple bending, 6 beams	Controlled by crushing of concrete	All beams: 1.10	0.069
Reinforced concrete rectangular columns	Axial loading, 16 columns	Controlled by crushing of concrete and yield of steel	All columns: 1.00	0.074
Reinforced concrete rectangular columns	Axial loading plus bending about one principal axis, 86 columns	Varied	All columns: 0.97	0.059
Reinforced concrete rectangular columns	Axial loading plus bending about two principal axes, 10 columns	Controlled by crushing of concrete and yield of some of the steel	All columns: 0.99	0.046
Reinforced concrete circular section columns	Eccentric load, 30 columns	Controlled by crushing of concrete and yield of some of the steel	All columns: 1.05	0.060

crete. The second assumption is therefore sufficiently accurate for the purpose of design calculations.

The maximum concrete compressive strain at ultimate strength has been measured in many tests of plain and reinforced concrete members. In Fig. 6a test results are plotted which were obtained at the University of Illinois[8,12] in tests of reinforced concrete columns and beams, while in Fig. 6b are shown values of maximum concrete compressive strain measured in the PCA[2] and Munich[11] tests of plain concrete. It can be seen that the value of 0.003 in. per in. proposed for design is a reasonably conservative value.

The implication of the fifth assumption is that the effect of strain hardening of the steel above the yield point is neglected. Compatibility of strains can no longer be relied on at high steel strains beyond the yield point, also the stress-strain curve beyond the yield point is not (and probably cannot conveniently become) stipulated in American specifications for manufacture of reinforcement. It is therefore considered unwise in design to rely on obtaining an increase in ultimate strength through strain hardening of the reinforcement.

SUMMARY OF COMPARISONS WITH EXPERIMENTS

Development of design equations and comparison with test data are presented in detail in Part 2 and the appendix, respectively. A summary is presented below.

The correctness of the equation for ultimate strength of a reinforced concrete beam reinforced only in tension, with the strength controlled by tension, is verified by R. C. Elstner's statistical analysis of 364 tests of this type of beam. This analysis leads to a value of 0.593 for the coefficient of q in the equation, as against a value of 0.59 derived from the assumptions set out in this paper, the differences being insignificant.

The comparisons of calculated and experimental ultimate strengths for other types of member and conditions of loading are summarized in Table 1. The data contained in the table were obtained from tests of 334 structural concrete elements of nine different types, in which the elements were subjected to four different combinations of flexure and axial loading. The tests included cases in which the concrete compression zone at ultimate strength was segmental, trapezoidal, or triangular in shape, in addition to the more common rectangular shape. It will be seen that there is close agreement between the experimental and calculated ultimate strengths. The mean value of "Test Ultimate Strength"/"Calculated Ultimate Strength" for all 334 cases considered is 1.037, and the standard deviation is 0.097.

CONCLUDING REMARKS

The validity of the basic assumptions, particularly the use of an equivalent rectangular stress distribution as proposed herein, and their applicability to the calculation of the ultimate strength of structural concrete sections has been confirmed by two findings:

1. The parameters proposed for use in ultimate strength design, as calculated from the basic assumptions, are in close agreement with the values of these parameters determined from tests of plain and reinforced concrete.

2. There is excellent agreement between the ultimate strengths of a wide range of structural concrete members determined experimentally, and the calculated ultimate strengths of the members based on the assumptions set out in this paper.

It is concluded, therefore, that the proposed method of ultimate strength design permits prediction with sufficient accuracy of the ultimate strength in bending, in compression, and in combinations of the two, of all types of structural concrete sections likely to be encountered in practice.

Part 2 — Design Equations

DESIGN OF RECTANGULAR BEAMS IN BENDING

Beams reinforced in tension only

Ultimate strength controlled by yielding of reinforcement—In this case, steel stress at ultimate strength, f_s, equals the yield stress f_y.

From equilibrium of internal forces in Fig. 7

$$0.85 k_1 f_c' \, b \, c = A_s \, f_y \qquad (1.1)$$

From equilibrium of internal and external moments

$$M_{ult} = A_s \, f_y (d - k_2 c) \qquad (1.2)$$

Solving Eq. (1.1) and (1.2) and substituting $q = p \, f_y / f_c'$:

$$M_{ult} = A_s \, f_y \, d \left(1 - \frac{k_2}{0.85 k_1} q \right) \qquad (1.3)$$

or

$$M_{ult} = b \, d^2 f_c' \, q \left(1 - \frac{k_2}{0.85 k_1} q \right) \qquad (1.3A)$$

Since $k_1 = 2k_2$, we may write

$$M_{ult} = A_s \, f_y \, d (1 - 0.59 q) \qquad (1.4)$$

or

$$M_{ult} = b \, d^2 f_c' \, q (1 - 0.59 q) \qquad (1.4A)$$

Eq. (1.4A) is, of course, identical with Eq. (A1) in the Appendix to ACI 318-56.

The correctness of Eq. (1.4A) was demonstrated statistically by R. C. Elstner. Using the general concrete stress distribution parameters

k_1, k_2, k_3, Eq. (1.3A) may also be written:

$$M_{ult} = bd^2 f_c' q \left(1 - \frac{k_2}{k_1 k_3} q \right) \quad (1.5)$$

Examining the results of 364 beam tests by the method of least squares k_2/k_1k_3 is given by:

$$\frac{k_2}{k_1 k_3} = \frac{- \Sigma \frac{M_{ult}}{bd^2 f_c'} q^2 + \Sigma q^3}{\Sigma q^4} \quad (1.6)$$

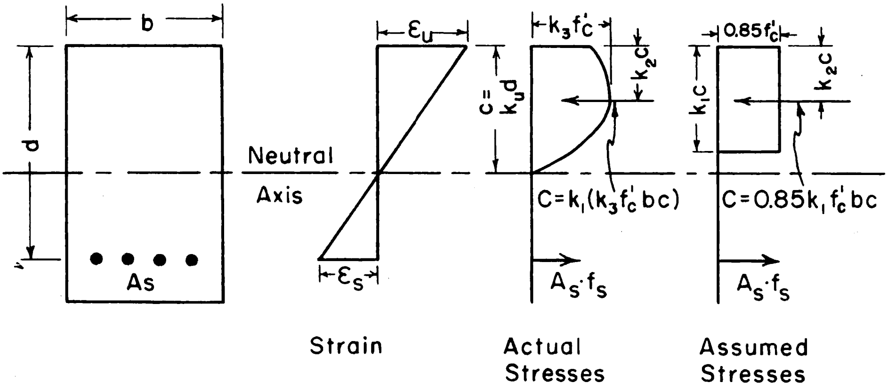


Fig. 7—Conditions at ultimate strength

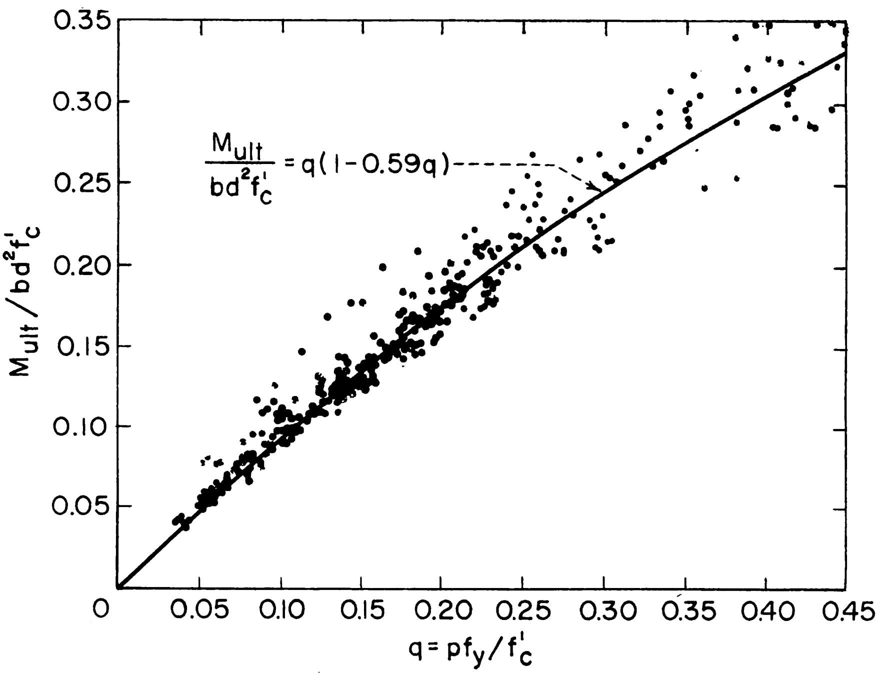


Fig. 8—Tests of 364 beams controlled by tension

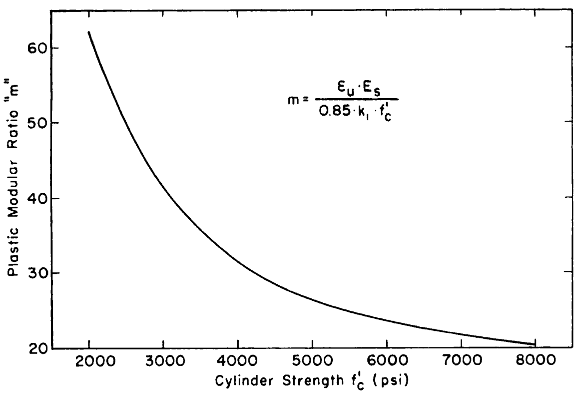


Fig. 9—Variation of plastic modular ratio *m* with concrete cylinder strength

Using this equation Elstner found $k_2/k_1k_3 = 0.593$. In Fig. 8, the curve of Eq. (1.4) is drawn, together with the 364 test results considered in Elstner's investigation.

Ultimate strength controlled by compression—In this case crushing of the concrete takes place while the steel stress is below the yield point.

From equilibrium of internal forces:

$$0.85\, k_1 f_c'\, b\, c = A_s f_s \qquad (1.1A)$$

From equilibrium of internal and external moments:

$$M_{ult} = A_s f_s (d - k_2 c) \qquad (1.2A)$$

From assumption of linear strain distribution:

$$k_u = \frac{c}{d} = \frac{\varepsilon_u}{\varepsilon_s + \varepsilon_u} \qquad (1.7)$$

Solving Eq. (1.1A) and (1.7) for k_u:

$$k_u = \sqrt{pm + \left(\frac{pm}{2}\right)^2} - \frac{pm}{2} \qquad (1.8)$$

where $m = (E_s\, \varepsilon_u) / (0.85\, k_1 f_c')$ as shown in Fig. 9.

The ultimate moment of resistance of a section may be calculated by substituting the value of k_u obtained from Eq. (1.8) in the following equation:

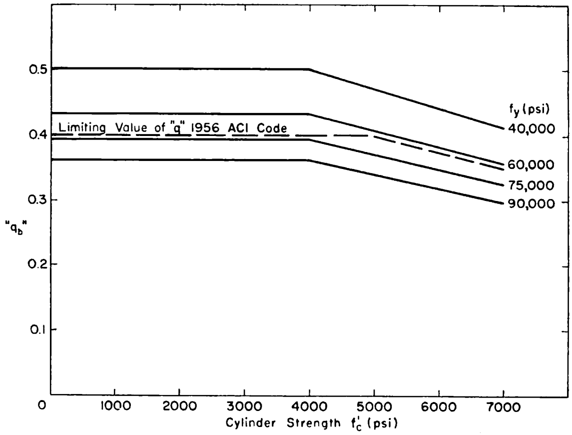
$$M_{ult} = (0.85\, k_1 f_c')\, b\, d^2 k_u (1 - k_2 k_u) \qquad (1.9)$$

Balanced conditions—In this case simultaneous yielding of steel and crushing of concrete takes place.

The balanced steel ratio, p_b, is obtained by solving Eq. (1.1) and (1.7):

$$p_b = 0.85 k_1 \frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u} \frac{f_c'}{f_y} \qquad (1.10)$$

Fig. 10—Steel ratio q_b for balanced ultimate strength



We may also write:

$$q_b = p_b \frac{f_y}{f_c'} = 0.85\, k_1 \frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u} \qquad (1.11)$$

or, substituting for ε_u and ε_y

$$q_b = 0.85\, k_1 \left(\frac{90{,}000}{f_y + 90{,}000}\right) \qquad (1.11A)$$

The ultimate moment for balanced failure may then be written:

$$M_{ult} = (0.85\, k_1 f_c')\, b\, d^2 \left(\frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u}\right)\left[1 - k_2\left(\frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u}\right)\right] \qquad (1.12)$$

Alternatively the value of q_b found using Eq. (1.11A) may be substituted in Eq. (1.4).

Fig. 10 shows the balanced steel ratios q_b for various strengths of concrete and steel. Also plotted in Fig. 10 is the limiting value of q to be used in design of sections prescribed by ACI 318-56. It is seen that the limiting value prescribed is safe for steel yield points not exceeding 60,000 psi as assumed in the Code, but the limit is in fact greater than q_b for high steel stresses. If it is considered desirable for design purposes to establish a limiting value of q less than q_b even for high steel stresses, then it is proposed that this limiting value be expressed as a fraction of q_b and not in the form in current use. If, however, a simple direct expression for the limiting value of q is desired, then the following is proposed:

"For $f_c' < 4000$ psi, $q_{lim} = 80/\sqrt{f_y}$; for concrete strengths greater than 4000 psi reduce q_{lim} by 0.02 for each 1000 psi." This simplified expression ensures that q_{lim} will be between 70 percent and 80 percent of q_b for an extremely wide range of concrete and steel strengths.

In Table A-1 of the appendix a comparison is made between the ultimate moment calculated using Eq. (1.9) and the ultimate moment measured in 59 tests on singly reinforced beams failing in compression. The average value of M_{test}/M_{calc} for all beams is 1.06 and the standard deviation is 0.113. It is of interest from a practical design point of view to note that the average value of M_{test}/M_{calc} for beams made from concrete with a cylinder strength of 2000 psi or over is 1.02, and that the standard deviation for this group of beams is 0.083. A histogram of these results is plotted in Fig. 11.

Beams reinforced in tension and compression

Ultimate strength controlled by yielding of tension reinforcement— In this case tensile steel stress at ultimate strength, f_s, is equal to the yield stress, f_y.

From equilibrium of internal forces in Fig. 12:

$$0.85\, k_1 f_c'\, b\, c + A_s' f_s' = A_s f_y \qquad (2.1)$$

From equilibrium of internal and external moments:

$$M_{ult} = 0.85\, k_1 f_c'\, b\, c(d - k_2 c) + A_s' f_s'(d - d') \qquad (2.2)$$

In most normal doubly reinforced beams the compression steel is close to the face of maximum concrete compression. The strain in the steel

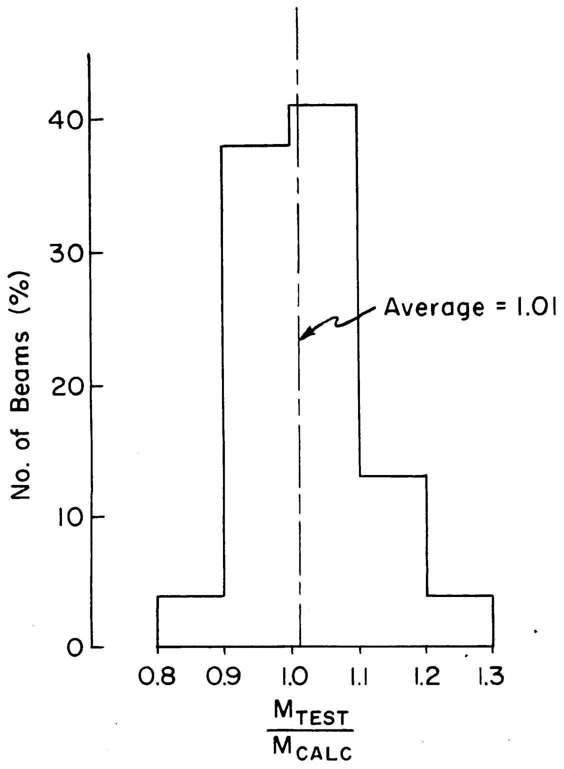


Fig. 11 — Histogram of (M_{test}/M_{calc}) for compression failures of singly reinforced concrete beams with concrete strength greater than 2000 psi

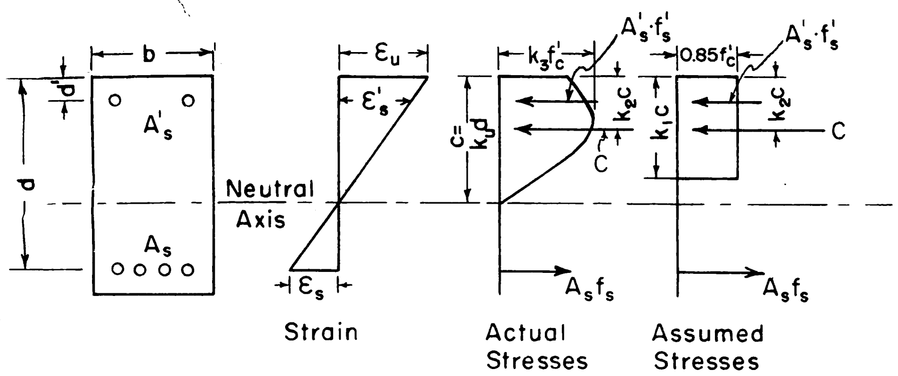


Fig. 12—Conditions at ultimate load, beam reinforced in tension and compression

will then be greater than the yield point strain ε_y', and the steel stress f_s' will equal f_y'. In this case we may rewrite Eq. (2.1) and (2.2) as follows:

$$0.85\, k_1 f_c'\, b\, c + A_s' f_y' = A_s f_y \quad \text{(2.1A)}$$

$$M_{ult} = 0.85\, k_1 f_c'\, b\, c\,(d - k_2 c) + A_s' f_y' (d - d') \quad \text{(2.2A)}$$

From Eq. (2.1A)

$$k_u = \frac{c}{d} = \left(\frac{f_y\, p - f_y'\, p'}{0.85\, k_1 f_c'}\right) \quad \text{(2.3)}$$

or

$$\frac{c}{d} = \left(\frac{q - q'}{0.85\, k_1}\right) \quad \text{(2.3A)}$$

where

$$q' = p' \frac{f_y'}{f_c'}$$

Calculate c from Eq. (2.3) and check that $f_s' = f_y'$ using

$$\varepsilon_s' = \varepsilon_u \frac{c - d'}{c} \quad \text{(2.4)}$$

If $\varepsilon_s' \geqq \varepsilon_y'$, the compression reinforcement has yielded and the ultimate moment may be calculated by solving Eq. (2.1A) and (2.2A), this yields:

$$M_{ult} = (A_s f_y - A_s' f_y')\, d \left\{ 1 - \frac{k_2}{0.85 k_1} \left(\frac{f_y\, p - f_y'\, p'}{f_c'} \right) \right\} + A_s' f_y' \{d - d'\}$$

Since $k_1 = 2k_2$, we may write:

$$M_{ult} = (A_s f_y - A_s' f_y')\, d \left\{ 1 - \frac{0.59}{f_c'} (f_y\, p - f_y'\, p') \right\} + A_s' f_y' \{d - d'\} \quad \text{(2.5)}$$

This may also be written:

$$M_{ult} = (A_s f_y - A_s' f_y')d\{1 - 0.59\,(q - q')\} + A_s' f_y' \{d - d'\} \quad \text{....(2.5A)}$$

If $f_y = f_y'$ this becomes

$$M_{ult} = (A_s - A_s') f_y\, d \left\{ 1 - 0.59 \frac{f_y}{f_c'}(p - p') \right\} + A_s' f_y \{d - d'\} \quad \text{......(2.5B)}$$

This is Eq. A3 of the Appendix to ACI 318-56, and is valid only if the compression steel has yielded. When using this equation a check should be made on the compression steel stress using Eq. (2.3) and (2.4).

If $\varepsilon_s' < \varepsilon_y'$, then compatibility of strains in the compression zone must be considered.

$$f_s' = \varepsilon_s' E_s = \varepsilon_u E_s \left(\frac{c - d'}{c}\right) \quad \text{......(2.6)}$$

Solving Eq. (2.1) and (2.6) for k_u:

$$k_u = \frac{c}{d} = \sqrt{\left[\frac{m}{2}\left(p' - \frac{\varepsilon_y}{\varepsilon_u} p\right)\right]^2 + p'm \frac{d'}{d}} - \frac{m}{2}\left(p' - \frac{\varepsilon_y}{\varepsilon_u} p\right) \quad \text{...(2.7)}$$

where, as before, $m = (E_s \varepsilon_u)/(0.85\, k_1 f_c')$.

The compression steel stress, f_s', may then be obtained from Eq. (2.6) by substituting the value of c, calculated using Eq. (2.7). The moment of resistance is obtained by substituting the calculated values of c and f_s' in Eq. (2.2).

Ultimate strength controlled by crushing of concrete—The tensile reinforcement is in this case stressed elastically. From equilibrium of internal forces:

$$0.85\, k_1 f_c' b\, c + A_s' f_s' = A_s f_s \quad \text{......(2.8)}$$

From equilibrium of internal and external moments:

$$M_{ult} = 0.85\, k_1 f_c' b\, c\, (d - k_2 c) + A_s' f_s' (d - d') \quad \text{......(2.9)}$$

If the compression reinforcement has yielded these equations become:

$$0.85\, k_1 f_c' b\, c + A_s' f_y' = A_s f_s \quad \text{......(2.8A)}$$

and

$$M_{ult} = 0.85\, k_1 f_c' b\, c\, (d - k_2 c) + A_s' f_y' (d - d') \quad \text{......(2.9A)}$$

From assumption of linear strain distribution:

$$k_u = \frac{c}{d} = \frac{\varepsilon_u}{\varepsilon_s + \varepsilon_u} \quad \text{......(2.10)}$$

or

$$\varepsilon_s = \varepsilon_u \left(\frac{d - c}{c}\right) \quad \text{......(2.10A)}$$

Solving Eq. (2.10) and (2.8A) for k_u

$$k_u = \frac{c}{d} = \sqrt{\left\{\frac{m}{2}\left(p + \frac{\varepsilon_y'}{\varepsilon_u}p'\right)\right\}^2 + pm} - \frac{m}{2}\left(p + \frac{\varepsilon_y'}{\varepsilon_u}p'\right) \quad (2.11)$$

where m is as before.

Check that ε_s' is $>$ ε_y' using $\varepsilon_s' = \varepsilon_u\,[(c - d')/c]$.

The moment of resistance is calculated using Eq. (2.9A), substituting the value of c, obtained from Eq. (2.11). If the compression reinforcement has not yielded, then the compatibility of strains across the entire section must be considered. This yields:

$$\varepsilon_s = \varepsilon_u \left(\frac{d - c}{c}\right) \quad (2.10A)$$

and

$$\varepsilon_s' = \varepsilon_u \left(\frac{c - d'}{c}\right) \quad (2.6)$$

Solving Eq. (2.10A), (2.6), and (2.8A) for k_u

$$k_u = \frac{c}{d} = \sqrt{\left\{\frac{m}{2}(p' + p)\right\}^2 + m\left(p'\frac{d'}{d} + p\right)} - \frac{m}{2}(p' + p) \quad (2.12)$$

The value of c obtained from Eq. (2.12) is substituted in Eq. (2.6) to give ε_s' and hence f_s'. These values of c and f_s' are substituted in Eq. (2.9) to yield the ultimate moment.

Balanced conditions—In this case simultaneous yielding of tensile steel and crushing of concrete takes place. From assumption of linear strain distribution:

$$c = \left(\frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u}\right) d \quad (2.13)$$

Solving Eq. (2.13) and (2.8):

Balanced steel ratio

$$p_B = p'\frac{f_s'}{f_y} + p_b \quad (2.14)$$

where p_b is the balanced steel ratio for a beam having the same steel and concrete strengths, but reinforced only in tension, as given by Eq. (1.10).

$$\varepsilon_s' = \varepsilon_u - \frac{d'}{d}(\varepsilon_y + \varepsilon_u)$$

If d'/d is $<$ $(\varepsilon_u - \varepsilon_y')/(\varepsilon_u + \varepsilon_y)$, then $\varepsilon_s' > \varepsilon_y'$ and $f_s' = f_y'$

$$p_B = p'\frac{f_y'}{f_y} + p_b \quad (2.15)$$

If d'/d is $> (\varepsilon_u - \varepsilon_y')/(\varepsilon_u + \varepsilon_y)$, then $f_s' < f_y'$

$$p_B = p' \frac{E_s}{f_y} \left\{ \varepsilon_u - \frac{d'}{d} (\varepsilon_y + \varepsilon_u) \right\} + p_b \qquad (2.16)$$

The ultimate moment at balanced failure is given by:

$$M_{ult} = 0.85\, k_1 f_c' bd^2 \left(\frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u}\right) \left\{ 1 - k_2 \left(\frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u}\right) \right\} + A_s' f_s' (d - d') \quad (2.17)$$

where

$$f_s' = f_y' \text{ if } \frac{d'}{d} \leq \left(\frac{\varepsilon_u - \varepsilon_y'}{\varepsilon_u + \varepsilon_y}\right),$$

and

$$f_s' = E_s \left\{ \varepsilon_u - \frac{d'}{d} (\varepsilon_y + \varepsilon_u) \right\} \text{ if } \frac{d'}{d} > \left(\frac{\varepsilon_u - \varepsilon_y'}{\varepsilon_u + \varepsilon_y}\right)$$

The results of applying the above analysis to 44 doubly reinforced beams is shown in detail in Table A-2 and is summarized below. (Histograms of these results are plotted in Fig. 13.)

Failure mode	Reinforcement	Average $\frac{M_{ult}\text{(test)}}{M_{ult}\text{(calc)}}$	Standard deviation
Tension	Structural or intermediate grade Steel	1.03	0.04
Tension	High strength alloy steel	1.21	0.17
Compression	Structural or intermediate grade Steel	1.03	0.07

It is of interest to note that the compression reinforcement yielded in only eight of the 33 beams failing in tension. In four of the beams the "compression" reinforcement was actually in tension at ultimate load.

The reason for the relatively high value of M_{test}/M_{calc} for beams failing in tension and reinforced with high strength alloy bars is probably strain hardening of the tension steel in beams with only a small percentage of tension reinforcement. The yield plateau of this reinforcement is of the order of 0.2 percent whereas that of structural grade reinforcement is about 1.5 percent. The alloy steel therefore quickly passes into the strain hardening range with consequently higher steel stresses at failure. Since neither the length of the yield plateau, nor the shape of the steel stress-strain curve beyond the yield plateau are included in steel specifications, it is the opinion of the writers that for practical design purposes the calculation of the ultimate moment of beams reinforced with high strength steel of this type should be based on the specified yield stress. However, if it is desired to analyze the

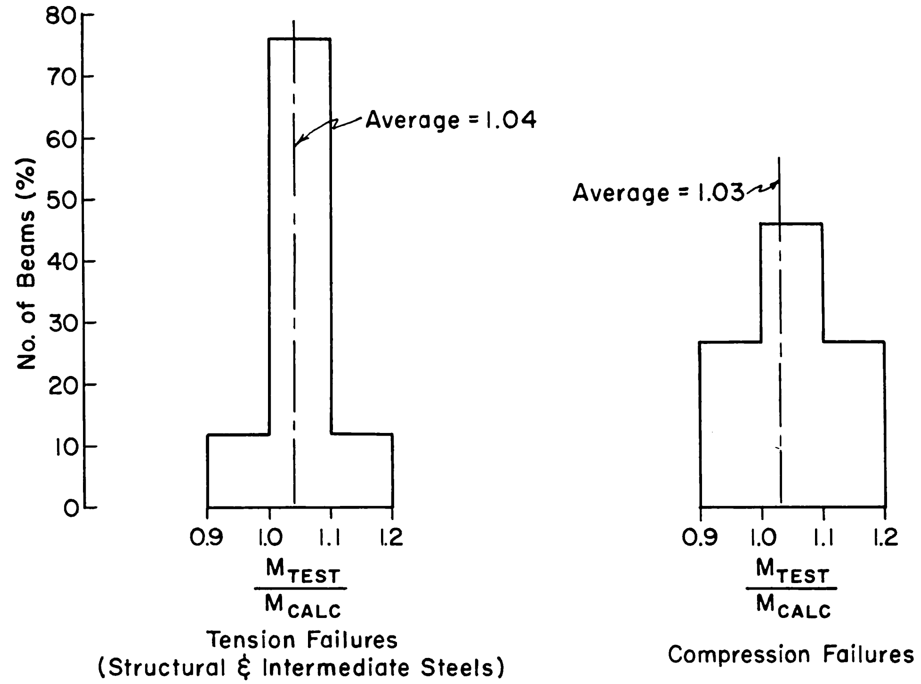


Fig. 13 — Histograms of (M_{test}/M_{calc}) for reinforced concrete beams with compression reinforcement

behavior of a specific beam, for which the stress-strain curve of the reinforcement is known, the following approach can be used.

From equilibrium of internal forces:

$$0.85\, k_1 f_c'\, b\, c + A_s' f_s' = A_s f_s \qquad (2.18)$$

From compatibility of strains:

$$\varepsilon_s' = \varepsilon_u \left(\frac{c - d'}{c}\right); \quad \text{and} \quad \frac{c}{d} = \left(\frac{\varepsilon_u}{\varepsilon_s + \varepsilon_u}\right)$$

Solving the above equations:

$$f_s = \left[\frac{0.85\, k_1 f_c'}{p}\left(\frac{\varepsilon_u}{\varepsilon_s + \varepsilon_u}\right) + \frac{p'}{p} E_s \varepsilon_u \left\{1 - \frac{d'}{d}\left(\frac{\varepsilon_s + \varepsilon_u}{\varepsilon_u}\right)\right\}\right] \qquad (2.19)$$

Plot the curve of f_s and ε_s given by Eq. (2.19) on the same base as the stress-strain curve for the reinforcement. The intersection of the curves gives the steel stress and strain at ultimate moment, from which the ultimate moment can be calculated. This approach was used to analyze Beam IIIB-1 from Table A-2, using a typical stress-strain curve for this type of reinforcement, and a value of M_{test}/M_{calc} of 1.23 was obtained, as against 1.44 if the calculated ultimate moment is based on the yield strength of the reinforcement. The remaining hyper-

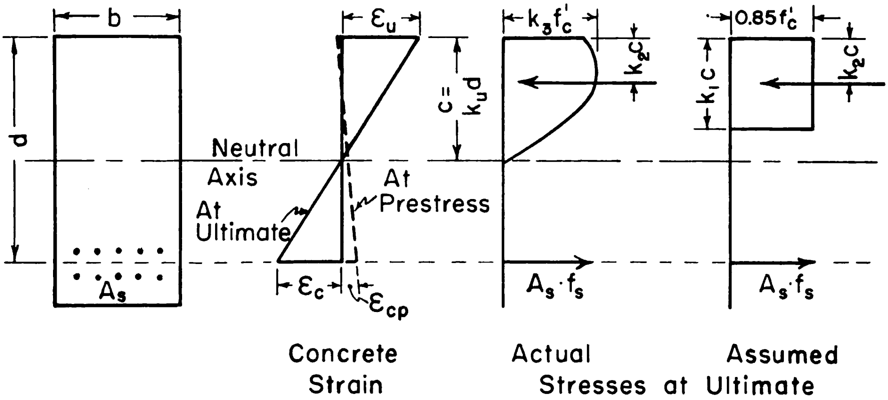


Fig. 14—Stress and strain conditions in prestressed beams

strength of 23 percent could be due to several as yet unpredictable causes such as variation in length of yield plateau from bar to bar, departure from linear strain relationship at a wide crack, etc.

Prestressed beams

The following reasoning applies to prestressed rectangular beams and to T- or I-section beams in which the flange thickness is greater than the depth of the equivalent rectangular stress block at failure.

From equilibrium of internal forces in Fig. 14:

$$0.85\, k_1 f_c' \, b \, c = A_s f_s \qquad (3.1)$$

From equilibrium of internal and external moments:

$$M_{ult} = A_s f_s \, (d - k_2 c) \qquad (3.2)$$

or

$$M_{ult} = 0.85\, k_1 f_c' \, b \, c \, (d - k_2 c) \qquad (3.2A)$$

From assumption of linear distribution of concrete strains:

$$k_u = \frac{c}{d} = \frac{\varepsilon_u}{\varepsilon_c + \varepsilon_u} \qquad (3.3)$$

Solving Eq. (3.1) and (3.3) we obtain:

(1) Change in concrete strain at level of steel from zero load to failure

$$(\varepsilon_c + \varepsilon_{cp}) = \varepsilon_{cp} + \varepsilon_u \left(\frac{0.85\, k_1 f_c'}{p f_s} - 1 \right) \qquad (3.4)$$

Because of high bond stresses and consequent bond slip at failure in this type of beam, the change in strain in the steel during loading to failure may not be equal to the change in strain in the adjacent concrete. We will write therefore:

(2) Change in strain in steel during loading to failure:

$$= \varepsilon_s - \varepsilon_{se} = F \left\{ \varepsilon_{cp} + \varepsilon_u \left(\frac{0.85\, k_1 f_c'}{p f_s} - 1 \right) \right\} \qquad (3.5)$$

where F is the ratio of the actual change in strain of the steel, to the change in strain of the concrete at the level of the steel calculated on the basis of linear distribution of concrete strain.

It has been suggested[23] that for well bonded tendons $F = 1.0$, while for post-tensioned tendons without bond a suitable value deduced from the experimental results[10,24,25,26] plotted in Fig. 15 is $F = 0.8k_u$. Therefore, for prestressed beams with well bonded tendons:

$$\varepsilon_s = \left\{ \varepsilon_u \left(\frac{0.85\, k_1 f_c'}{p f_s} - 1 \right) + \varepsilon_{cp} + \varepsilon_{se} \right\} \qquad (3.6)$$

while for prestressed beams with unbonded tendons:

$$\varepsilon_s = \left\{ \frac{0.8 p f_s}{0.85\, k_1 f_c'} (\varepsilon_{cp} - \varepsilon_u) + 0.8\, \varepsilon_u + \varepsilon_{se} \right\} \qquad (3.7)$$

Since the shapes of the stress-strain curves of prestressing steel do not in general lend themselves to algebraic representation, the analysis of a given section is best carried out using a process of iteration. Assuming a value of f_s, the steel stress at failure, the corresponding strain ε_s may be calculated using Eq. (3.6) or (3.7). The steel strain so obtained is entered on the stress-strain diagram for the reinforcing steel and a new value of f_s obtained. The process is repeated, using the new value of f_s, as often as is necessary.

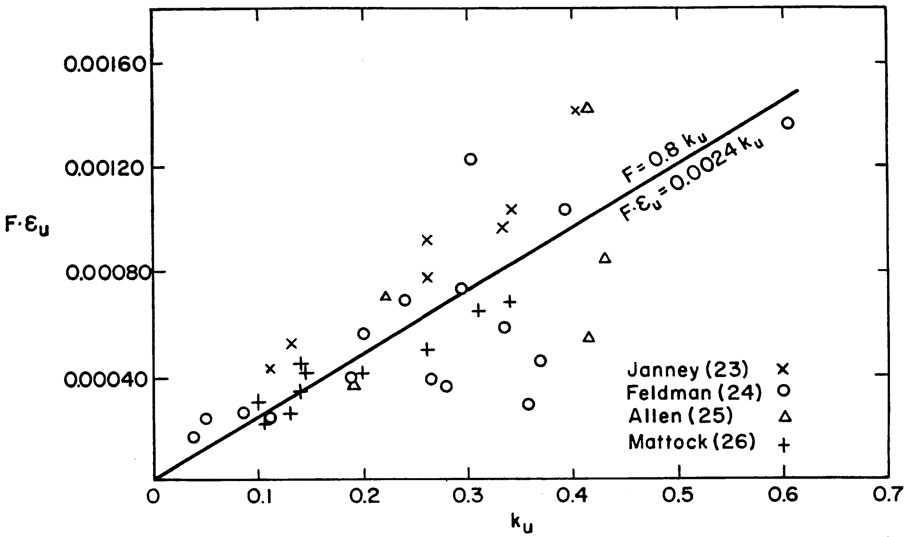


Fig. 15—Post-tensioned prestressed beams without bond $F \cdot \varepsilon_u$ v k_u

When the steel stress has been determined, the neutral axis depth may be calculated using Eq. (3.1) and the ultimate moment using Eq. (3.2).

Value of ε_{cp}: For a safe estimate of ε_s, the minimum probable value of ε_{cp} should be used. It can be shown[26] that with present day allowable concrete stresses a reasonable value of ε_{cp} is 0.0004.

The analysis proposed above was applied to 32 prestressed beams with well bonded tendons and to 24 post-tensioned beams without bond; the results are detailed in Tables A-4 and A-5 and are summarized below.

1. Beams with well bonded tendons:
   Average M_{test}/M_{calc} = 1.07; Standard deviation = 0.077
2. Post-tensioned beams without bond:
   Average M_{test}/M_{calc} = 1.05; Standard deviation = 0.094

Histograms of these results are plotted in Fig. 16.

Note — Expressions for the strength of prestressed beams with non-rectangular compression zones may be derived by combining the approach used above to take into account the influence of prestress, with the approach described below for reinforced concrete beams with nonrectangular compression zones.

NONRECTANGULAR BEAMS

Symmetrical T-beams

Ultimate strength controlled by yielding of reinforcement — When the flange thickness exceeds the depth of the equivalent rectangular stress block, the equations developed for the singly reinforced rectangu-

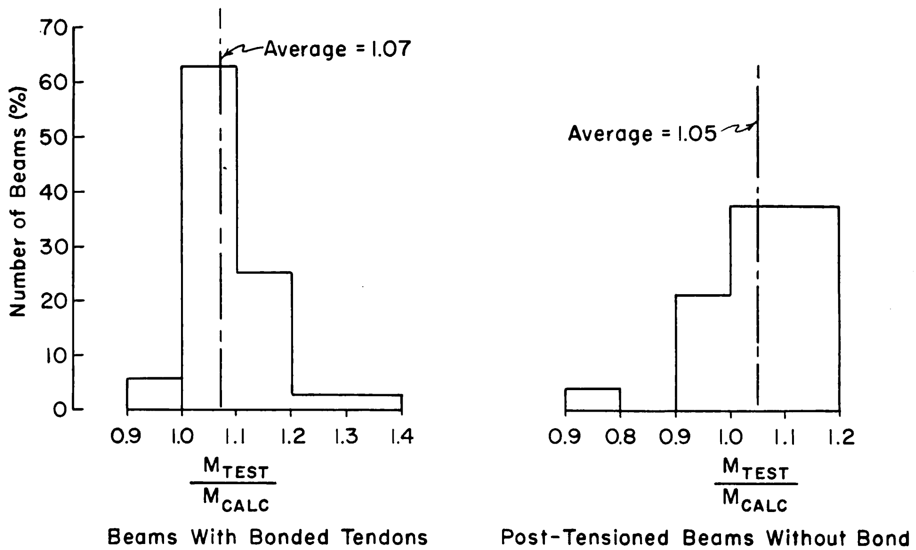


Fig. 16 — Histograms of (M_{test}/M_{calc}) for prestressed beams with and without bond

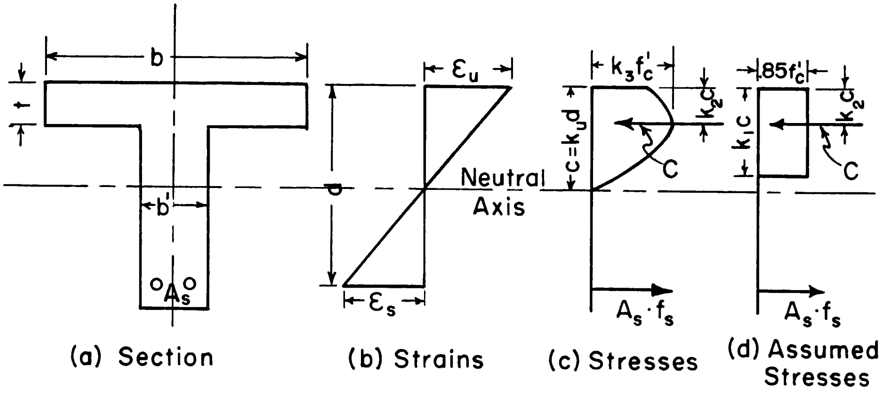


Fig. 17—Conditions at ultimate strength in reinforced concrete T-beam

lar beams are valid for the T-beam. If the depth of the equivalent rectangular stress block is greater than the flange thickness, i.e., if

$$k_1c = (A_s f_y)/(0.85 f_c' b) > t \text{ (Fig. 17), then:}$$

From equilibrium of internal forces:

$$0.85\, k_1 f_c' b' c + 0.85\, f_c' (b - b') t = A_s f_y \quad \text{......(4.1)}$$

From equilibrium of internal and external moments:

$$M_{ult} = 0.85\, k_1 f_c' b' c \left(d - \frac{k_1}{2} c \right) + 0.85 f_c' (b - b') t \left(d - \frac{t}{2}\right) \quad \text{....(4.2)}$$

Solving Eq. (4.1) and (4.2):

$$M_{ult} = (A_s - A_{sf}) f_y d \left\{ 1 - 0.59 \frac{f_y}{b'd f_c'} (A_s - A_{sf}) \right\} + A_{sf} f_y \left(d - \frac{t}{2}\right) \quad (4.3)$$

In which $A_{sf} = \{0.85 f_c' (b - b')t\}/f_y$ is the steel area necessary to develop the compressive strength of the overhanging portions of the flange. Eq. (4.3) may also be written as follows:

$$M_{ult} = (A_s - A_{sf}) f_y d \left\{ 1 - 0.59 (p_w - p_f) \frac{f_y}{f_c'} \right\} + A_{sf} f_y \left(d - \frac{t}{2}\right) \quad \text{......(4.4)}$$

where: $p_w = A_s/b'd$ and $p_f = A_{sf}/b'd$. This is Eq. (A4) of ACI 318-56.

Ultimate strength controlled by crushing of concrete—While steel stress is still below the yield point:

From equilibruim of internal forces:

$$0.85\, k_1 f_c' b' c + 0.85 f_c' (b - b') t = A_s f_s \quad \text{......(4.1A)}$$

From equilibrium of internal and external moments:

$$M_{ult} = 0.85\, k_1 f_c' b' c \left(d - \frac{k_1}{2} c \right) + 0.85 f_c' (b - b') t \left(d - \frac{t}{2}\right) \quad \text{....(4.2)}$$

From assumption of linear strain distribution:

$$k_u = \frac{c}{d} = \frac{\varepsilon_u}{\varepsilon_s + \varepsilon_u} \quad \text{(4.5)}$$

Solving Eq. (4.1A) and (4.5) for k_u:

$$k_u = \frac{c}{d} = \sqrt{p_w\, m + \left[\left\{\left(\frac{b-b'}{b'}\right)\frac{t}{2\,k_1\,d}\right\} + \frac{p_w\, m}{2}\right]^2} - \left[\left\{\left(\frac{b-b'}{b'}\right)\frac{t}{2\,k_1\,d}\right\} + \frac{p_w\, m}{2}\right] \quad \text{(4.6)}$$

where, as before, $m = (E_s\,\varepsilon_u)/(0.85k_1f_c')$.

The ultimate moment of resistance of the section may be calculated by substituting the value of c obtained from Eq. (4.6) in Eq. (4.2) above.

Balanced conditions—i.e., simultaneous yielding of steel and crushing of concrete.

From assumption of linear strain distribution:

$$\frac{c}{d} = \left(\frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y}\right) \quad \text{(4.7)}$$

Solving Eq. (4.1) and (4.7):

$$\left(\frac{A_s - A_{sf}}{b'\,d}\right) = 0.85\,k_1\,\frac{f_c'}{f_y}\left(\frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u}\right) = p_b \quad \text{(4.8)}$$

Where p_b is the balanced steel ratio for a rectangular beam made of same concrete and steel.

The ultimate moment for balanced conditions may be written:

$$M_{ult} = 0.85\,k_1\,f_c'\,b'\left(\frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y}\right)d^2\left\{1 - \frac{k_1}{2}\left(\frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y}\right)\right\} + 0.85\,f_c'\,(b-b')\,t\left(d - \frac{t}{2}\right) \quad \text{(4.9)}$$

The results of tests on 15 T-beams were analyzed and are detailed in Table A-3. The mean value of M_{test}/M_{calc} is 1.11 and the standard deviation 0.10. The results include some beams in which strain hardening of the reinforcement is known to have occurred. If desired the strain hardening effect could be investigated in a particular case using the approach outlined for doubly reinforced beams.

Nonsymmetrical T-beams

For most nonsymmetrical T-beams used in monolithically cast construction, lateral deflection is prevented. The neutral axis is then horizontal, and the strength of an unsymmetrical T-beam equals that of an equivalent symmetrical beam with the same total flange width.

When lateral restraint is absent, however, the beam may be considered to be loaded vertically and to be free to deflect laterally as shown in Fig. 18. Since no moments are applied in a horizontal plane, the center of concrete compression must be vertically above the centroid of the tension reinforcement. The width of the stress block at the top face of the beam will therefore be 1:5 times the width of the web of the beam for this condition to be satisfied, assuming the reinforcement is placed symmetrically within the web.

Ultimate strength controlled by yield of reinforcement—From equilibrium of internal forces:

$$\tfrac{1}{2}\,(1.5\,b'\,a\,0.85\,f_c') = A_s\,f_y \qquad (5.1)$$

From equilibrium of internal and external moments:

$$M_{ult} = A_s\,f_y\left(d - \frac{a}{3}\right) \qquad (5.2)$$

or

$$= 0.75\,b'\,a\,0.85\,f_c'\left(d - \frac{a}{3}\right) \qquad (5.2A)$$

Solving Eq. (5.1) and (5.2) we have:

$$M_{ult} = A_s\,f_y\,d\,(1 - 0.52\,q_w) \qquad (5.3)$$

or

$$M_{ult} = b'\,d^2 f_c'\,q_w\,(1 - 0.52\,q_w) \qquad (5.3A)$$

where

$$q_w = \left(\frac{A_s}{b'\,d}\,\frac{f_y}{f_c'}\right)$$

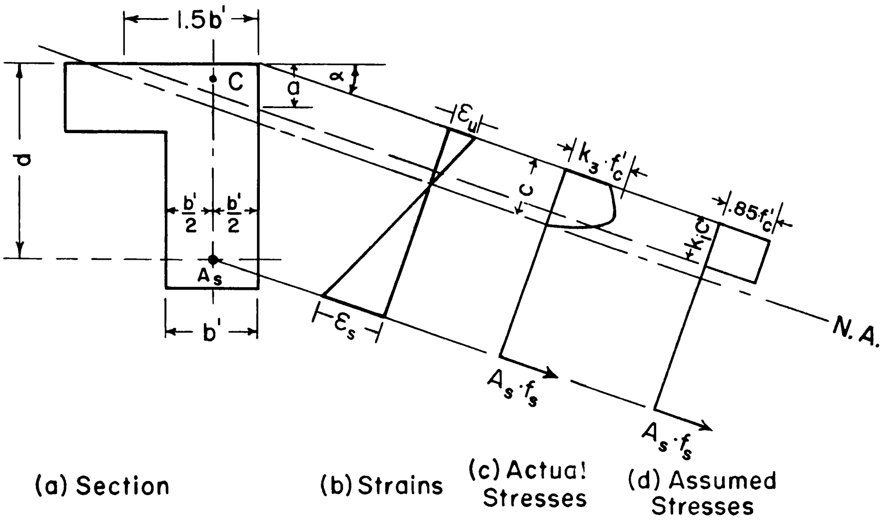


Fig. 18—Conditions at ultimate strength in a reinforced concrete unsymmetrical T-beam

Ultimate strength controlled by crushing of concrete—While the steel stress is below the yield point.

From the geometry of the section, as shown in Fig. 18:

$$c = \frac{3}{2\,k_1} b' \sin \alpha \qquad (5.4)$$

where α is the inclination of the neutral axis to the horizontal.

From the assumption of linear strain distribution:

$$\frac{\varepsilon_s}{\varepsilon_u} = \frac{(d \cos \alpha + \tfrac{1}{2} b' \sin \alpha - c)}{c} \qquad (5.5)$$

From equilibrium of internal forces:

$$\tfrac{1}{2} (\tfrac{3}{2} b')^2 \tan \alpha \; 0.85 f_c' = A_s E_s \varepsilon_s \qquad (5.6)$$

From equilibrium of internal and external moments:

$$M_{ult} = \tfrac{1}{2} (\tfrac{3}{2} b')^2 \tan \alpha \; 0.85 f_c' (d - \tfrac{1}{2} b' \tan \alpha) \qquad (5.7)$$

Solving Eq. (5.4), (5.5), and (5.6) for tan α we have:

$$\tan \alpha = \left[\frac{\sqrt{\left(1 - \frac{k_1}{3}\right)^2 + \left(\frac{3\, b'\, d}{m\, A_s}\right)} - \left(1 - \frac{k_1}{3}\right)}{\left(\frac{9\,(b')^2}{4\, k_1\, m\, A_s}\right)} \right] \qquad (5.8)$$

where, as before, $m = (\varepsilon_u E_s) / (0.85 k_1 f_c')$.

The ultimate moment is calculated by substituting the value of tan α from Eq. (5.8) in Eq. (5.7).

Balanced condition—Simultaneous crushing of concrete and yield of steel.

From equilibrium of internal forces:

$$\tfrac{1}{2} (\tfrac{3}{2} b')^2 \tan \alpha \; 0.85 f_c' = A_s f_y = A_s E_s \varepsilon_y \qquad (5.9)$$

From assumption of linear strain distribution:

$$\frac{\varepsilon_y}{\varepsilon_u} = \frac{(d \cos \alpha + \tfrac{1}{2} b' \sin \alpha) - \left(\frac{3}{2\,k_1} b' \sin \alpha\right)}{\left(\frac{3}{2\,k_1} b' \sin \alpha\right)}$$

$$\therefore \frac{\varepsilon_y}{\varepsilon_u} = \frac{2\,k_1}{3}\left(\frac{d}{b'} \cot \alpha + \tfrac{1}{2}\right) - 1 \qquad (5.10)$$

Solving Eq. (5.9) and (5.10) for the steel ratio p_{wb} for balanced ultimate conditions:

$$p_{wb} = \frac{f_c'}{f_y} \left[\frac{1.91\, k_1}{3\left(\frac{\varepsilon_y}{\varepsilon_u} + 1\right) - k_1} \right] \qquad (5.11)$$

or

$$q_{wb} = \frac{1.91\, k_1}{3\left(\frac{\varepsilon_y}{\varepsilon_u} + 1\right) - k_1} \quad \text{(5.11A)}$$

The ultimate moment of resistance is given by substituting q_{wb} in Eq. (5.3)

$$M_{ult} = A_s f_y d\,(1 - 0.52\, q_{wb}) \quad \text{(5.12)}$$

The results of tests on 12 unsymmetric T-beams are summarized in Table A-6. The mean value of M_{test}/M_{calc} for all beams is 1.10, and the standard deviation is 0.138. The mean value of M_{test}/M_{calc} for beams with ultimate strength controlled by tension is 1.04 and the standard deviation is 0.047.

Beams with triangular compression zones

The following analysis was developed in connection with tests of this type of beam, to check the applicability of the rectangular stress block approximation to members having triangular shaped compression zones.

Ultimate strength controlled by yield of reinforcement—From equilibrium of internal forces (see Fig. 19)

$$0.85\, f_c'\, a^2 \tan\theta = A_s f_y \quad \text{(6.1)}$$

From equilibrium of internal and external moments:

$$M_{ult} = A_s f_y \left(d - \frac{2a}{3}\right) \quad \text{(6.2)}$$

Solving Eq. (6.1) and (6.2) we have:

$$M_{ult} = A_s f_y d\,(1 - 0.723\sqrt{q}) \quad \text{(6.3)}$$

or

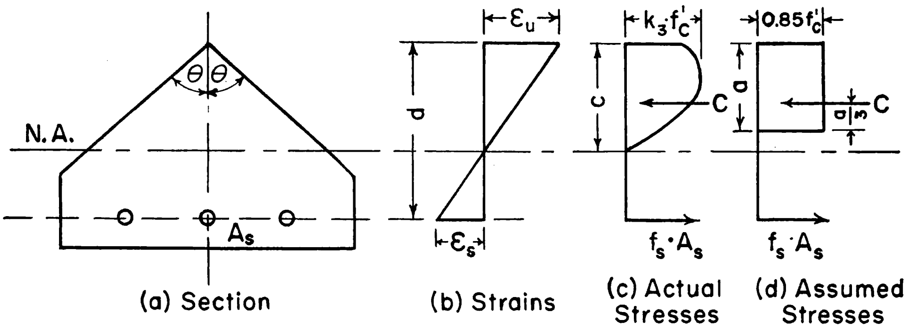


Fig. 19—Condition at ultimate strength in a reinforced concrete beam with a triangular shaped compression zone

$$M_{ult} = d^3 \tan \theta \, f_c' \, q \, (1 - 0.723 \sqrt{q}) \qquad (6.3A)$$

where

$$q = \left(\frac{A_s}{d^2 \tan \theta} \, \frac{f_y}{f_c'} \right)$$

Ultimate strength controlled by crushing of the concrete—While the steel stress is below the yield point.

From equilibrium of internal forces:

$$0.85 \, f_c' \, a^2 \tan \theta = A_s f_s \qquad (6.4)$$

From equilibrium of internal and external moments:

$$M_{ult} = 0.85 \, f_c' \, a^2 \tan \theta \left(d - \frac{2a}{3} \right) \qquad (6.5)$$

From assumption of linear strain distribution:

$$\frac{\varepsilon_s}{\varepsilon_u} = \left(\frac{d - c}{c} \right) \qquad (6.6)$$

From Eq. (6.4) and (6.6):

$$c^3 \left(\frac{0.85 \, f_c' \, k_1^2 \tan \theta}{A_s \, E_s \, \varepsilon_u} \right) + c - d = 0 \qquad (6.7)$$

Solve Eq. (6.7) for c and substitute in Eq. (6.5A) below to compute ultimate moment

$$M_{ult} = 0.85 \, f_c' \, k_1^2 \, c^2 \tan \theta \left(d - \frac{2k_1}{3} c \right) \qquad (6.5A)$$

Alternatively Eq. (6.4) and (6.6) can be solved for c using an iterative procedure. The value of c obtained is then, as before, substituted in Eq. (6.5A).

Balanced conditions—Simultaneous crushing of concrete and yielding of steel.

From equilibrium of internal forces:

$$0.85 \, f_c' \, (k_1 c)^2 \tan \theta = A_s f_y \qquad (6.8)$$

From assumption of linear strain distribution:

$$\frac{c}{d} = \left(\frac{\varepsilon_u}{\varepsilon_y + \varepsilon_u} \right) \qquad (6.9)$$

Solving Eq. (6.8) and (6.9) for the steel ratio p_b for balanced ultimate conditions $[p = A_s / (d^2 \tan \theta)]$.

$$p_b = \frac{0.85 \, f_c'}{f_y} \left(\frac{k_1 \, \varepsilon_u}{\varepsilon_y + \varepsilon_u} \right)^2 \qquad (6.10)$$

or

$$q_b = 0.85 \left(\frac{k_1 \varepsilon_u}{\varepsilon_y + \varepsilon_u} \right)^2 \qquad (6.10A)$$

The ultimate moment of resistance is given by substituting q_b in Eq. (6.3)

$$M_{ult} = A_s f_y d \, (1 - 0.723 \sqrt{q_b}) \qquad (6.11)$$

The results of tests on six beams with triangular compression zone are summarized in Table A-7. The mean value of M_{test}/M_{calc} is 1.10 and the standard deviation is 0.069.

DESIGN OF COLUMNS

Concentrically loaded columns

In this case it is assumed that at ultimate strength of a column the entire concrete cross-section is uniformly stressed to 85 percent of the cylinder strength of the concrete, and that the entire cross section of the column is uniformly strained to 0.003 in. per in.

For columns reinforced with steels having a yield point of 90,000 psi or less, ultimate strength will be controlled by simultaneous crushing of the concrete and yielding of the steel. The ultimate strength is therefore given by:

$$P_o = 0.85 f_c' A_c + f_y A_{st} \qquad (7.1)$$

where A_c is the net concrete cross section
A_{st} is the total longitudinal reinforcement cross section

In the above it is assumed that sufficient lateral ties are provided to ensure that the reinforcement will not buckle before reaching its yield stress.

Eq. (7.1) was used to calculate the ultimate strength of 16 concentrically loaded reinforced concrete columns with lateral ties, tested as part of the ACI column investigation and reported in University of Illinois, *Engineering Experiment Station Bulletin* No. 267.[6] The results are set out in detail in Table A-8. The average value of P_{test}/P_{calc} for all beams is 1.00, and the standard deviation is 0.074.

It was concluded in the ACI column investigation[6] that Eq. (7.1) also serves to calculate the yield point load of a concentrically loaded circular column with spiral reinforcement. Further increase in load is obtainable in this type of column when loaded concentrically due to the lateral restraint of the compressed concrete by the spiral. However, Hognestad[8] has shown that if the load is eccentric to even a small degree, then no further increase in load is obtained after yield in this type of column. Since a truly concentrically loaded column is an extreme rarity in practice, it is proposed that increase in load after yield of

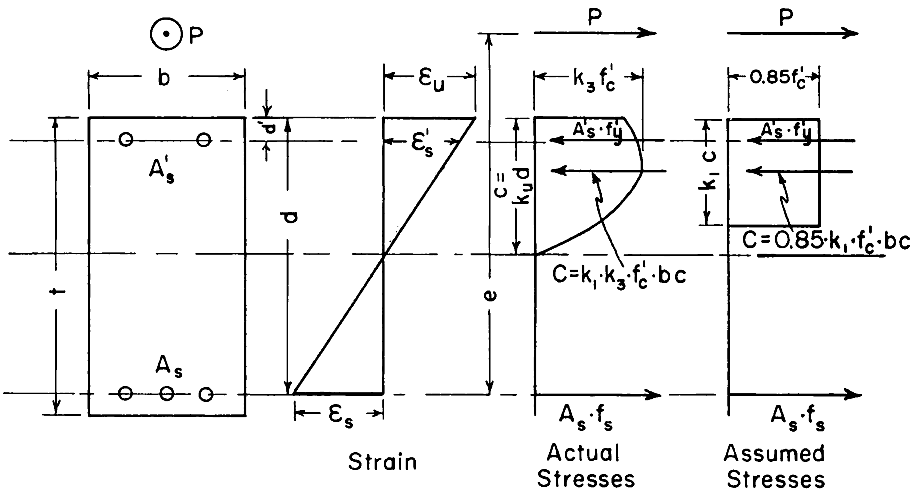


Figure 20—Conditions in an eccentrically loaded column at failure, $c < d$

spirally reinforced columns should be ignored, and that their ultimate strength should be calculated using Eq. (7.1).

Eccentrically loaded rectangular column, reinforced on two faces

It is reasonable to assume that the compression reinforcement has yielded. This may be verified by checking that $\varepsilon_u = [(c - d')/c]$ is $\geqq \varepsilon_y'$. If this is not the case, a solution may be developed by taking into account compatibility of strains, as was done for the case of a beam reinforced in compression in which the compression reinforcement did not yield.

In what follows it is assumed that the above check has been made, and that the compression reinforcement has been found to yield.

From equilibrium of internal and external forces in Fig. 20:

$$P_u = 0.85\, k_1 f_c'\, b\, c + A_s' f_y' - A_s f_s \qquad (8.1)$$

From equilibrium of internal and external moments:

$$P_u e = 0.85\, k_1 f_c'\, b\, c\, (d - k_2 c) + A_s' f_y' (d - d') \qquad (8.2)$$

Strength governed by tension steel yielding—Solving Eq. (8.1) and (8.2) for c/d, and substituting

$$q = \frac{p f_y}{f_c'} \quad \text{and} \quad q' = \frac{p' f_y'}{f_c'}$$

$$\frac{c}{d} = \frac{1}{k_1}\left[- \left(\frac{e}{d} - 1\right) + \sqrt{\left\{\frac{e}{d} - 1\right\}^2 - 2.35 \left\{ q' \left(\frac{e + d'}{d} - 1\right) - q\left(\frac{e}{d}\right)\right\}}\,\right] \qquad (8.3)$$

Substituting this value of c/d in Eq. (8.1) yields:

$$P_u = f_c' b d \left[(q' - q) + 0.85 \left\{ -\left(\frac{e}{d} - 1\right) + \sqrt{\left(\frac{e}{d} - 1\right)^2 - 2.35\, q' \left(\frac{e + d'}{d} - 1\right) - q \frac{e}{d}} \right\} \right] \quad \text{......(8.4)}$$

For symmetrically reinforced columns, $p = p'$. Assuming $f_y = f_y'$, $q = q'$, therefore:

$$P_u = 0.85 f_c' b d \left[-\left(\frac{e}{d} - 1\right) + \sqrt{\left(\frac{e}{d} - 1\right)^2 + 2.35\, q \left(\frac{d - d'}{d}\right)} \right] \quad \text{....(8.5)}$$

In the above equations the displacement of concrete by the compression reinforcement is ignored. For values of $p' > 2.0$ percent the ultimate load may be over-estimated by up to 6 percent.[8] This error may be corrected by considering an effective yield point $[f_y' - 0.85 f_c']$ instead of f_y' in the calculation q'.

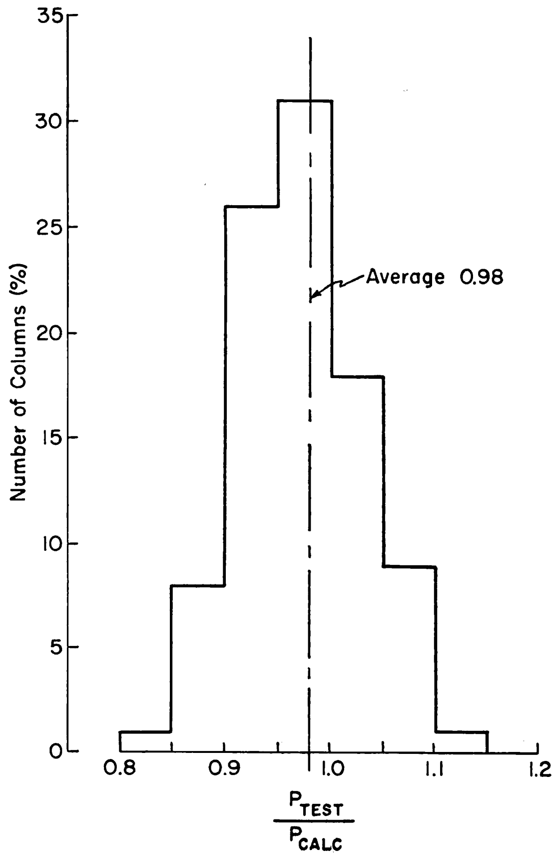


Fig. 21 — Histogram of (P_{test}/P_{calc}) for eccentrically loaded rectangular columns

Strength governed by concrete crushing—From assumption of linear distribution of strains:

$$\frac{c}{d} = \left(\frac{\varepsilon_u}{\varepsilon_u + \varepsilon_s}\right) \qquad (8.6)$$

and,

$$f_s = E_s\, \varepsilon_s = E_s\, \varepsilon_u \left(\frac{d - c}{c}\right) \qquad (8.6A)$$

If $k_1\, c$ is less than t, Eq. (8.1) and (8.2) apply.

If $k_1\, c$ is greater than t, these equations become:

$$P_u = 0.85\, f_c'\, b\, t + A_s'\, f_y' + A_s\, f_s \qquad (8.1A)$$

$$P_u e = 0.85\, f_c'\, b\, t \left(d - \frac{t}{2}\right) + A_s'\, f_y'\, (d - d') \qquad (8.2A)$$

Once again, if greater precision is desired in computation of the ultimate load, an effective yield point $[f_y' - 0.85\, f_c']$ should be used instead of f_y'.

Solving Eq. (8.1), (8.2), and (8.6A) yields a cubic equation in c/d which does not lend itself to easy solution.

The following method of solution by successive approximations was proposed by Hognestad.[8]

With known section dimensions, material constants, and eccentricity e, the problem is to find c, and hence P_u. The procedure is as follows. Assume a value for c and use Eq. (8.2) to calculate P_u. Substituting the assumed value for c in Eq. (8.6A) calculate f_s. Using the calculated values of P_u and f_s in Eq. (8.1) calculate c. The process is repeated until the assumed and final value of "c" coincide or are sufficiently close.

Balanced conditions — Simultaneous yielding of tension steel and crushing of concrete.

In this case $\varepsilon_s = \varepsilon_y$

$$\therefore \frac{c}{d} = \left(\frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y}\right) \qquad (8.7)$$

Solving Eq. (8.1) and (8.7):

$$P_u = 0.85\, k_1\, f_c'\, b\, d \left(\frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y}\right) + A_s'\, f_y' - A_s\, f_y \qquad (8.8)$$

Solving Eq. (8.2) and (8.7):

$$P_u e = 0.85\, k_1\, f_c'\, b\, d^2 \left(\frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y}\right) \left\{ 1 - k_2 \left(\frac{\varepsilon_u}{\varepsilon_u + \varepsilon_y}\right) \right\} + A_s'\, f_y\, (d - d') \qquad (8.9)$$

The analysis proposed above was applied to 84 eccentrically loaded, rectangular reinforced concrete columns tested by Hognestad.[8] The results are set out in detail in Table A-9. The average value of P_{test}/P_{calc} for all columns is 0.97, and the standard deviation is 0.059. A histogram

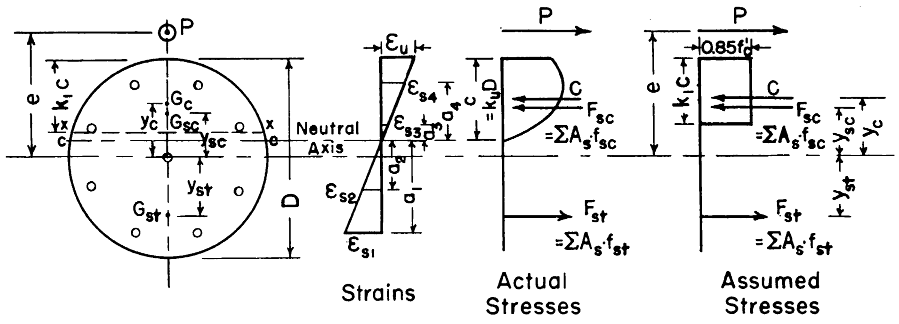


Fig. 22—Stress and strain conditions in circular column subject to eccentric loading

of these results is plotted in Fig. 21. The calculation of the ultimate strength of this large number of columns was greatly facilitated by use of an electronic computer.

Circular columns subject to eccentric load

Case 1—Equivalent rectangular stress distribution covers part only of column cross section as shown in Fig. 22.

Let G_{sc} be center of action of steel compressive force.
Let G_{st} be center of action of steel tensile force.
Let G_c be center of action of concrete compressive force.

From condition of equilibrium of internal and external forces:

$$P_u = 0.85 f_c' A_c + F_{sc} - F_{st} \qquad (9.1)$$

where

A_c = area of segment of circle covered by equivalent rectangular stress distribution.

F_{sc} = resultant steel compressive force $(\Sigma A_s f_{sc})$

F_{st} = resultant steel tensile force $(\Sigma A_s f_{st})$

From condition of equilibrium of internal and external moments:

$$P_u e = 0.85 f_c' A_c y_c + F_{sc} y_{sc} + F_{st} y_{st} \qquad (9.2)$$

Where the eccentricity e is measured from the center of the column.

For the general case an iterative solution of the Eq. (9.1) and (9.2) is proposed.

Assume a position of the neutral axis cc and calculate the stress in each bar of the reinforcement using:

$$f_{sn} = E_s \varepsilon_{sn} = E_s \varepsilon_u \left(\frac{a_n}{c}\right) \text{ ; which must be } < f_y$$

Calculate the resultant tensile and compressive steel forces, F_{st} and F_{sc}, and their centers of action, $G_{st\,(y_{st})}$; $G_{sc\,(y_{sc})}$.

The first moment of area about center O of the segmental area covered by the rectangular stress distribution, $A_c y_c$, is given by:

$$A_c\, y_c = \frac{(\overline{xx})^3}{12} = 2/3\ \{ D\,(k_1\, c) - (k_1\, c)^2 \}^{3/2} \qquad (9.3)$$

Substitute the values calculated above in Eq. (9.2) and obtain P_u.

Substitute this value of P_u together with the calculated values of F_{sc} and F_{st} in Eq. (9.1) to obtain A_c, and hence a value for k_1c and c.

The calculation should be repeated as often as is necessary, until assumed and final values of c are equal or sufficiently close. The value of P_u corresponding to this value of c is the ultimate load capacity of the column.

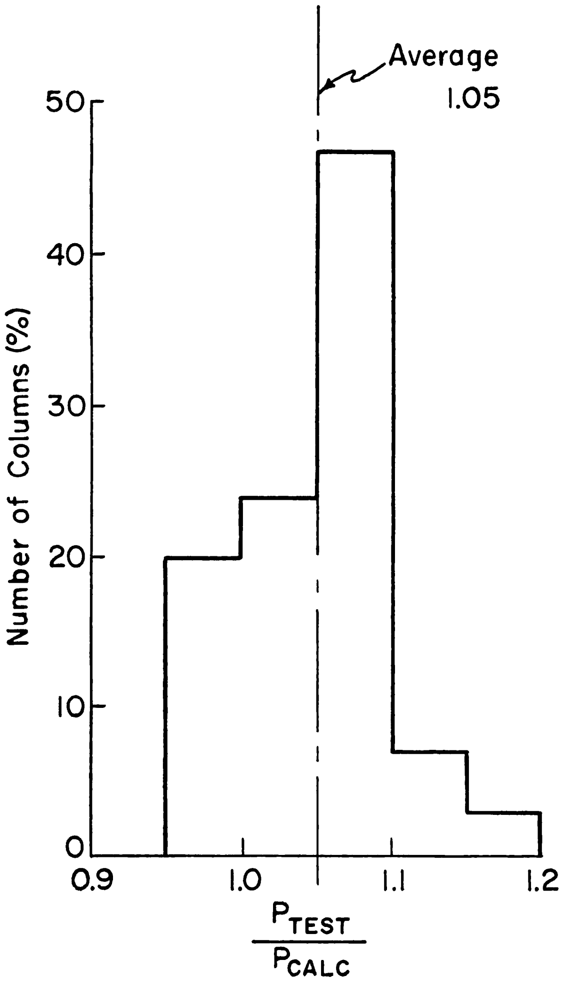


Fig. 23 — Histogram of (P_{test}/P_{calc}) for 30 eccentrically loaded circular columns

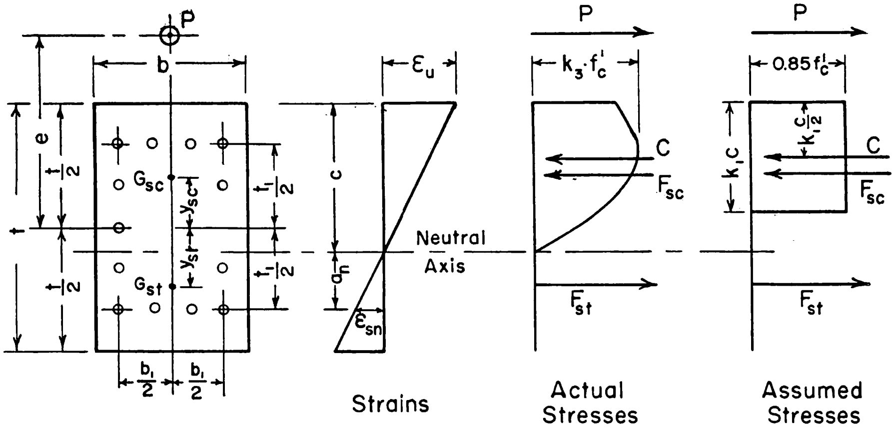


Fig. 24—Conditions at ultimate load, eccentrically loaded rectangular column with reinforcement on all four faces

Case 2—Equivalent rectangular stress distribution covers the entire column cross section.

From condition of equilibrium of internal and external forces:

$$P_u = 0.85 f_c' \left(\frac{\pi}{4} D^2\right) + F_{sc} \quad \text{.......... (9.4)}$$

From condition of equilibrium of internal and external moments:

$$P_u e = F_{sc} y_{sc} \quad \text{.......... (9.5)}$$

As before assume a position for the neutral axis and calculate F_{sc} and y_{sc}. Using these values of F_{sc} and y_{sc} in Eq. (9.5) calculate P_u. From Eq. (9.4) calculate F_{sc} inserting value of P_u found above.

Adjust the assumed neutral axis position until the two values of F_{sc} calculated are equal or sufficiently close.

The above method of analysis was applied to 30 eccentrically loaded circular columns tested by Hognestad[8] and the results are contained in Table A-10. The average of P_{test}/P_{calc} for the whole series was 1.05, and the standard deviation 0.060. A histogram of the results is plotted in Fig. 23.

Rectangular columns, reinforced at all four faces (Fig. 24)

Let G_{sc} be center of action of steel compressive force.

Let G_{st} be center of action of steel tensile force.

Let F_{sc} be resultant steel compressive force $(\Sigma A_s f_{sc})$.

Let F_{st} be resultant steel tensile force $(\Sigma A_s f_{st})$.

From condition of equilibrium of internal and external forces:

$$P_u = 0.85 f_c' b k_1 c + F_{sc} - F_{st} \quad \text{.......... (10.1)}$$

From condition of equilibrium of internal and external moments:

$$P_u e = 0.85 f_c' b k_1 c \left(\frac{t}{2} - k_1 \frac{c}{2}\right) + F_{sc} y_{sc} + F_{st} y_{st} \quad \text{........(10.2)}$$

In the general case an iterative solution of the above equations is proposed.

Assume a position of the neutral axis and calculate the stress in each bar of the reinforcement using:

$$f_{sn} = E_s \varepsilon_{sn} = E_s \varepsilon_u \left(\frac{a_n}{c}\right) \text{; which must be } \leq f_y$$

Calculate the resultant steel forces F_{sc} and F_{st}, and their centers of action G_{sc} and G_{st}. With these values of F_{sc} and F_{st}, calculate P_u using Eq. (10.1). Substitute this value of P_u in Eq. (10.2) together with the previously calculated values of F_{sc}, F_{st}, y_{sc}, y_{st}, and hence calculate the neutral axis depth c.

The calculation should be repeated as often as is necessary until the assumed and final values of c coincide or are sufficiently close. The value of P_u corresponding to this value of c is the ultimate load capacity of the column for the particular eccentricity considered.

Rectangular columns with bending on both principal axes (Fig. 25)

Let G_{sc} be center of action of steel compressive force.
Let G_{st} be center of action of steel tensile force.
Let G_c be center of action of concrete compressive force.

From condition of equilibrium of internal and external forces:

$$P_u = 0.85 f_c' A_c + F_{sc} - F_{st} \quad \text{........(11.1)}$$

where

A_c = area of compression zone covered by rectangular stress block
F_{sc} = resultant steel compressive force ($\Sigma A_s f_{sc}$)
F_{st} = resultant steel tensile force ($\Sigma A_s f_{st}$)

From condition of equilibrium of internal and external moments:

$$0.85 f_c' A_c x_c = (F_{st} x_{st} - F_{sc} x_{sc} - P_u e_x) \quad \text{........(11.2)}$$

$$0.85 f_c' A_c y_c = (F_{st} y_{st} - F_{sc} y_{sc} - P_u e_y) \quad \text{........(11.3)}$$

In the general case an iterative solution of the above three equations is proposed.

Assume a position of the neutral axis cc and calculate the stress in each bar of the reinforcement using:

$$f_{sn} = E_s \varepsilon_{sn} = E_s \varepsilon_u \left(\frac{a_n}{c}\right) \text{; which must be } \leq f_y$$

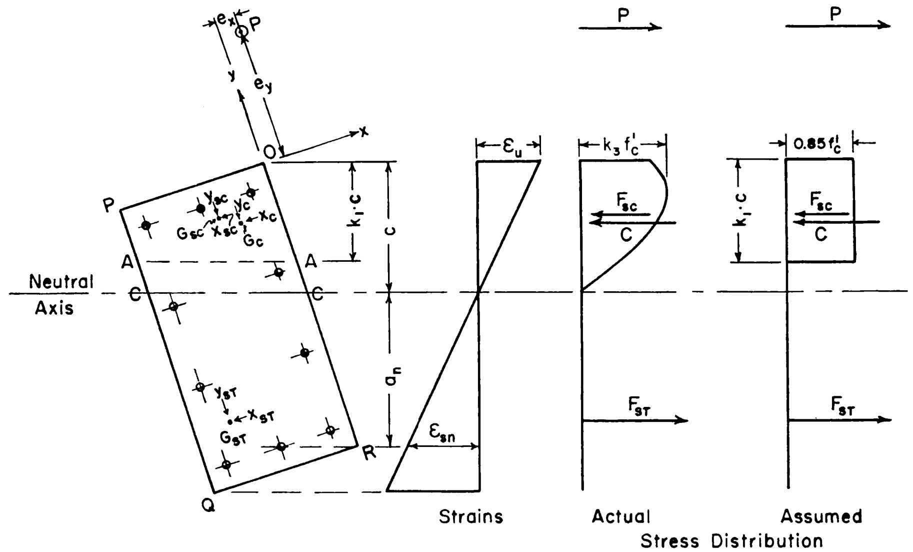


Fig. 25—Conditions at ultimate load in column subject to axial load and bendind on both principal axes

Calculate the resultant tensile and compressive steel forces, F_{st} and F_{sc}, and their centers of action $G_{st\,(x_{st},\,y_{st})}$ and $G_{sc\,(x_{sc},\,y_{sc})}$. Using the above values of F_{st} and F_{sc}, find P_u for the assumed neutral axis position using Eq. (11.1). From Eq. (11.2) and (11.3) obtain $A_c y_c$ and $A_c x_c$, using the values of P_u, F_{sc}, etc. already found.

It is then possible to calculate the dimensions of the area covered by the rectangular stress distribution and hence to obtain the position of the neutral axis. The calculation must be repeated as often as is necessary until assumed and calculated positions of neutral axis coincide or are sufficiently close. The value of P_u corresponding to this neutral axis position is the ultimate strength of the column.

The calculation of the dimensions of the compression zone from values of $A_c y_c$ and $A_c x_c$ obtained as outlined above is considered below.

Case 1—Line AA cuts adjacent sides OP and OR as in Fig. 26.

$$A_c x_c = \frac{X^2 Y}{6} \qquad A_c y_c = \frac{X Y^2}{6}$$

$$\therefore X = \sqrt[3]{\frac{6(A_c x_c)^2}{(A_c y_c)}}$$

and

$$Y = \left(\frac{6 A_c x_c}{X^2}\right)$$

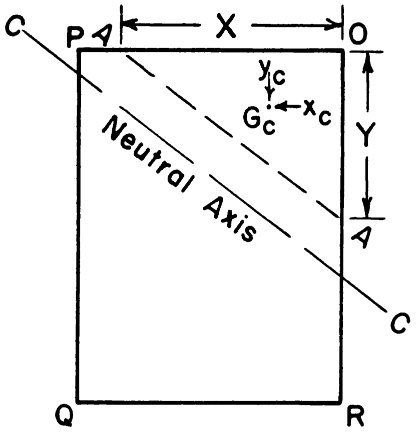


Fig. 26 — Geometry of compression zone, Case I

Case 2—Line *AA* cuts opposite sides *PQ* and *OR* as in Fig. 27.

$$A_c x_c = \frac{b^2}{2}\left[Y_1 + \frac{Y_2}{3}\right]$$

$$A_c y_c = \frac{b}{2}\left[Y_1^2 + Y_2\left(Y_1 + \frac{Y_2}{3}\right)\right]$$

$$\therefore\ Y_1 = \left[3\left(\frac{A_c x_c}{b^2}\right) - \sqrt{\frac{2}{b}(A_c y_c) - 3\left(\frac{A_c x_c}{b^2}\right)^2}\ \right]$$

and

$$Y_2 = 3\left[\frac{2}{b^2} A_c x_c - Y_1\right]$$

Case 3—Line *AA* cuts opposite sides of *OP* and *QR* as in Fig. 28.

$$A_c x_c = \frac{t}{2}\left[X_1^2 + X_2\left(X_1 + \frac{X_2}{3}\right)\right]$$

$$A_c y_c = \frac{t^2}{2}\left[X_1 + \frac{X_2}{3}\right]$$

$$\therefore\ X_1 = \left[3\left(\frac{A_c y_c}{t^2}\right) - \sqrt{\frac{2}{t}\ (A_c x_c) - 3\ \left(\frac{A_c y_c}{t^2}\right)}\right]$$

$$X_2 = 3\left[\frac{2}{t^2}\ A_c y_c - X_1\right]$$

Case 4—Line *AA* cuts adjacent sides *PQ* and *QR* as in Fig. 29. Use *Q* as origin for x, y, and e.

$$P_u e_x - F_{sc} x_{sc} + F_{st} x_{st} = 0.85 f_c' A_c x_c \quad \text{(11.2A)}$$

$$P_u e_y - F_{sc} y_{sc} + F_{st} y_{st} = 0.85 f_c' A_c y_c \quad \text{(11.3A)}$$

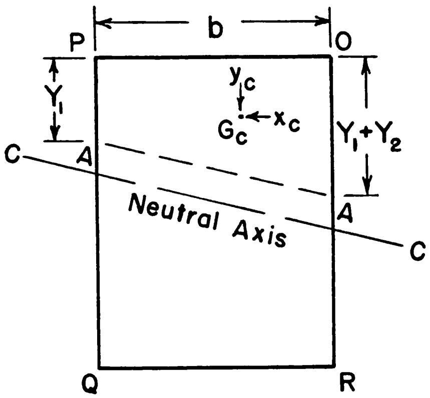


Fig. 27 — Geometry of compression zone, Case 2

$$A_c x_c = \left(\frac{b^2 t}{2} - \frac{X_1^2 Y_1}{6}\right)$$

$$A_c y_c = \left(\frac{b\, t^2}{2} - \frac{X_1 Y_1^2}{6}\right)$$

$$\therefore\ Y_1 = \frac{6}{X_1^2}\left[\frac{b^2\, t}{2} - A_c x_c\right]$$

$$X_1 = \sqrt[3]{\frac{6\left[\frac{b^2\, t}{2} - A_c x_c\right]^2}{\left[\frac{b\, t^2}{2} - A_c y_c\right]}}$$

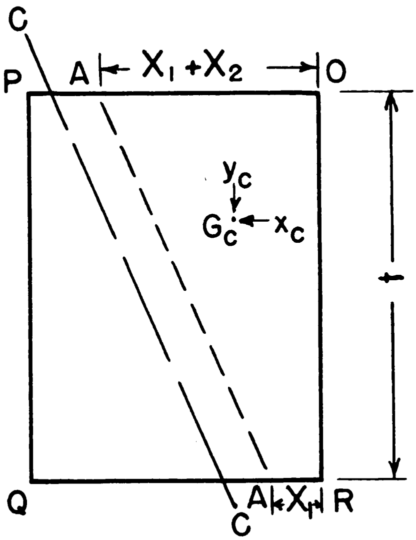


Fig. 28 — Geometry of compression zone, Case 3

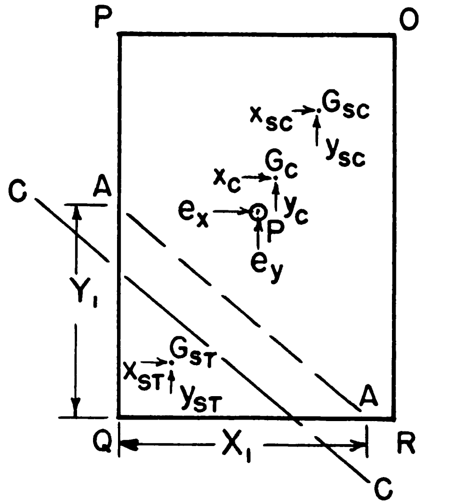


Fig. 29 — Geometry of compression zone, Case 4

Case 5—Equivalent rectangular stress distribution covers entire cross section. As in Case 4, use Q as origin. Let neutral axis CC intersect x and y axes at X' and Y' distance x' and y' from Q.

$$\text{Again } f_{sn} = E_s \varepsilon_u \left(\frac{a_n}{c}\right), \text{ which must be } \leqq f_y.$$

Assume position of neutral axis. Calculate stress in each bar, and hence total steel force F_s and point of action $G_s(x_s\, y_s)$. Then:

$$P_u = F_s + 0.85\, f_c'\, b\, t \qquad (11.4)$$

From condition of equilibrium of internal and external moments:

$$P_u(e_x) = \{0.85\, f_c'\, b\, t\}\, \frac{b}{2} + F_s\, x_s \qquad (11.5)$$

and

$$P_u(e_y) = \{0.85\, f_c'\, b\, t\}\, \frac{t}{2} + F_s\, y_s \qquad (11.6)$$

Substituting the value of P_u from Eq. (11.4), Eq. (11.5) and (11.6) may be solved for x_s and y_s. If these values of x_s and y_s do not agree with the values calculated above using the assumed position of neutral axis, a new position of neutral axis must be assumed and the process be repeated as often as is necessary. In a simple section it may be possible to link x_s and y_s to x' and y' algebraically, in which case iteration may be used to solve the problem rather than trial and error.

Test results for columns subject to axial load and bending about both principal axes are rather scarce. The analysis proposed has been applied to ten columns tested by Andersen and Lee[27] and the results are contained in Table A-11. The average value of P_{test}/P_{calc} for the series is 0.99, and the standard deviation is 0.046.

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APPENDIX — TEST DATA

TABLE A-1—REINFORCED CONCRETE BEAMS REINFORCED IN TENSION ONLY (Ultimate strength controlled by compression)

Source	Beam No.	f_c', psi	f_y, ksi	p	$\frac{M_{ult}}{bd^2 f_c'}$ Test	$\frac{M_{ult}}{bd^2 f_c'}$ Calculated	Test / Calculated
W. A. Slater and I. Lyse[13]	1	1390	64.8	0.021	0.506	0.362	1.40
	2	2790	64.8	0.028	0.337	0.342	0.99
	3	4070	64.8	0.037	0.326	0.336	0.97
	4	4800	64.8	0.047	0.339	0.333	1.02
	5	5740	64.8	0.056	0.320	0.323	0.99
	6	2590	64.8	0.030	0.422	0.349	1.21
	6A	4130	64.8	0.039	0.327	0.337	0.97
	7	2950	64.8	0.028	0.341	0.339	1.01
	8	2760	64.8	0.031	0.380	0.348	1.09
	9	2820	64.8	0.032	0.391	0.347	1.13
	10	2820	64.8	0.030	0.346	0.345	1.00
	10A	3810	64.8	0.040	0.354	0.344	1.03
Columbia University[14]	C1	3550	61.37	0.0341	0.406	0.339	1.20
	C11	3550	62.00	0.0345	0.394	0.340	1.16
	C2	3550	63.04	0.0334	0.386	0.338	1.14
	C12	3550	64.28	0.0328	0.365	0.337	1.08

TABLE A-1 (cont.) — REINFORCED CONCRETE BEAMS REINFORCED IN TENSION ONLY (Ultimate strength controlled by compression)

Source	Beam No.	f_c', psi	f_y, ksi	p	$\frac{M_{ult}}{bd^2 f_c'}$ Test	$\frac{M_{ult}}{bd^2 f_c'}$ Calculated	$\frac{\text{Test}}{\text{Calculated}}$
K. C. Cox[15]	122	1700	53.4	0.0176	0.405	0.343	1.18
	123	1700	53.4	0.0264	0.423	0.363	1.16
	124	1700	53.4	0.0352	0.455	0.374	1.22
	125	1700	53.4	0.0440	0.471	0.382	1.23
	142	1700	48.1	0.0244	0.442	0.359	1.23
	143	1700	48.1	0.0368	0.426	0.376	1.13
	144	1700	48.1	0.0488	0.490	0.385	1.27
	224	3100	53.4	0.0352	0.343	0.348	0.98
	225	3100	53.4	0.0440	0.353	0.359	0.98
	243	3100	48.1	0.0368	0.370	0.350	1.06
	244	3100	48.1	0.0488	0.375	0.363	1.03
	214	3100	55.2	0.0308	0.374	0.341	1.10
	215	3100	55.2	0.0388	0.372	0.353	1.05
	235	3100	48.1	0.0388	0.381	0.353	1.08
	252	3100	50.6	0.0348	0.379	0.348	1.09
	253	3100	50.6	0.0520	0.388	0.366	1.06
	325	4500	53.4	0.0440	0.320	0.336	0.95
	343	4500	48.1	0.0368	0.297	0.326	0.91
	344	4500	48.1	0.0488	0.342	0.341	1.00
	425	5800	53.4	0.0440	0.293	0.309	0.94
	444	5800	48.1	0.0488	0.300	0.314	0.95
S. L. Lash and J. W. Brison[16]	4205	1970	39.2	0.0400	0.449	0.374	1.20
	4206	1930	44.4	0.0475	0.464	0.380	1.22
	4308	3330	42.8	0.0454	0.367	0.357	1.03
	4407	4170	40.8	0.0367	0.305	0.333	0.91
	4408	4490	43.4	0.0471	0.346	0.339	1.02
	6203	2150	88.0	0.0140	0.346	0.316	1.09
	6204	2150	75.8	0.0200	0.333	0.338	0.99
	6205	1950	75.8	0.0225	0.415	0.349	1.19
	6206	2080	73.5	0.0284	0.400	0.357	1.12
	6207	1915	75.2	0.0385	0.391	0.373	1.05
	6208	2120	75.2	0.0391	0.407	0.370	1.10
	6303	3290	72.0	0.0147	0.270	0.292	0.92
	6304	2760	75.8	0.0233	0.347	0.332	1.04
	6305	3200	74.0	0.0286	0.338	0.335	1.01
	6306	2760	75.2	0.0394	0.359	0.359	1.00
	6404	4490	75.8	0.0226	0.263	0.296	0.89
	6405	4140	74.0	0.0280	0.293	0.318	0.92
	6406	4190	75.2	0.0390	0.317	0.336	0.94
	6407	4190	62.1	0.0408	0.314	0.338	0.93
	6504	4870	75.8	0.0233	0.247	0.289	0.85
	6505	4450	65.0	0.0371	0.299	0.327	0.91
	6506	5450	75.8	0.0458	0.292	0.318	0.92

Avg, all beams (59) 1.06
Standard deviation, all beams .... ±0.113

Avg for beams with $f_c' > 2000$ psi 1.02
Standard deviation for beams with f_c' 2000 psi 0.083

TABLE A-2—REINFORCED CONCRETE BEAMS REINFORCED IN TENSION AND COMPRESSION

Source	Beam	b	d	d'	A_s	A_s'	p, per-cent	p', per-cent	f_c', psi	f_y, ksi	f_y', ksi	$\frac{M}{bd^2f_c'}$ Test	$\frac{M}{bd^2f_c'}$ Calc	Test/Calc	Failure modes*
Gaston[17]†	c2w	6	10.58	0.90	0.88	0.40	1.38	0.63	3940	45.4	44.5	0.153	0.148	1.03	2
	c2xm	6	10.58	1.29	0.88	0.40	1.38	0.63	4070	53.3	47.0	0.175	0.163	1.07	2
(tension	c3w	6	10.37	1.48	2.00	1.20	3.22	1.93	4310	41.8	46.7	0.275	0.275	1.00	2
failures)	c3xm	6	10.37	1.48	2.00	1.20	3.22	1.93	3890	41.8	42.5	0.323	0.303	1.07	2†
	c3yna	6	10.51	1.41	1.20	0.62	1.90	0.98	3330	45.2	56.1	0.236	0.230	1.03	2
	c3ynb	6	10.37	1.55	2.00	1.20	3.22	1.93	4860	42.1	47.4	0.242	0.245	0.99	2
	c4xna	6	10.51	1.22	1.20	0.62	1.90	0.98	2450	45.5	41.4	0.310	0.313	0.99	1†
	c4xnb	6	10.51	1.51	1.20	0.88	1.90	1.39	2430	46.4	44.1	0.324	0.319	1.02	2
	c4zn	6	10.37	1.48	2.00	1.20	3.22	1.93	3570	41.3	46.4	0.329	0.326	1.01	2
	c5yn	6	9.38	1.57	3.16	1.58	5.61	2.80	4480	44.0	43.4	0.457	0.463	0.99	1†
	c6xm	6	9.38	1.57	3.16	1.58	5.61	2.80	3680	41.8	40.2	0.531	0.526	1.01	1†
	c7w	6	9.38	1.57	3.16	1.58	5.61	2.80	3480	41.6	43.6	0.552	0.552	1.00	1†
McCol-lister[18]‡	B-34	6	10.58	1.40	0.88	0.40	1.39	0.63	4760	46.5	47.2	0.139	0.125	1.00	2
	B-35	6	10.51	1.41	1.20	0.62	1.90	0.98	2345	48.4	47.2	0.347	0.343	1.01	1†
	T-1	6	10.58	1.28	0.88	0.40	1.39	0.63	3897	41.8	47.9	0.148	0.138	1.07	2
(tension	T-2	6	10.37	1.51	2.00	1.20	3.21	1.93	3858	45.4	50.0	0.334	0.330	1.01	2
failures)	T-3	6	10.20	1.56	3.12	1.58	5.10	2.58	4266	44.7	46.1	0.455	0.453	1.01	1†
	T-7	6	10.58	1.42	0.88	0.88	1.39	1.39	4540	40.9	42.5	0.127	0.116	1.09	2
	T-10	6	10.58	1.33	0.88	0.22	1.39	0.35	4330	42.6	46.3	0.142	0.126	1.13	2
	T-12	6	10.28	1.84	2.54	2.00	4.12	3.24	4367	46.0	45.8	0.390	0.369	1.06	2
	T-14	6	10.58	1.52	0.88	0.88	1.39	1.39	4030	41.4	40.9	0.145	0.131	1.00	2
	T-15	6	10.58	1.50	0.88	0.88	1.39	1.39	3700	47.0	40.5	0.171	0.159	1.07	2
	T-4	6	10.65	1.28	0.62	0.40	0.97	0.63	2230	47.1	45.7	0.190	0.186	1.02	2
	T-5	6	10.51	1.35	1.20	0.62	1.90	0.98	2021	48.4	46.6	0.407	0.396	1.03	1†
	T-6	6	10.37	1.49	2.00	1.20	3.21	1.93	1905	56.5	49.7	0.802	0.790	1.01	2
Gural-nick[19]	IB-1	23	11.81	2.31	1.20	1.20	0.442	0.442	2440	84.6	84.6	0.157	0.135	1.16	2
	IIB-1	23	11.81	2.31	1.20	1.20	0.442	0.442	2440	84.6	84.6	0.197	0.135	1.46	2
	IID-1	23	12.05	2.81	2.08	2.08	0.750	0.750	5520	87.7	87.7	0.133	0.112	1.19	2
(tension	IIIA-1	23	11.81	2.74	2.08	2.08	0.765	0.765	3130	87.7	87.7	0.216	0.188	1.15	2
failures)	IIIB-1	23	11.81	2.31	1.20	1.20	0.442	0.442	3130	84.6	84.6	0.161	0.112	1.44	2
	IIIB-2	23	12.12	2.63	1.20	1.20	0.430	0.430	3130	84.6	84.6	0.116	0.110	1.05	2
	IIIC-1	23	11.78	2.68	3.73	3.73	1.380	1.380	5300	83.9	83.9	0.203	0.207	0.98	2
	IIID-1	23	11.81	2.72	2.08	2.08	0.766	0.766	5300	87.7	87.7	0.159	0.125	1.27	2

John-son[20]	2	4.02	5.40	0.67		0.089	4.6	0.41	2860	96.0	53.8	0.503	0.425	1.18	3
	6	3.94	5.28	0.67		0.089	4.6	0.43	2820	96.0	51.1	0.442	0.426	1.04	3
	10	4.02	5.28	0.94		0.089	4.6	0.42	3200	96.0	51.1	0.427	0.408	1.05	3
(Com-pression failures)	14	3.98	5.28	0.67		0.089	4.6	0.42	2830	96.0	51.5	0.435	0.438	0.99	3
	3	4.14	5.12	0.71		0.244	4.6	1.15	3240	96.0	46.1	0.497	0.488	1.02	3
	7	3.98	5.28	0.67		0.244	4.6	1.16	2780	96.0	46.1	0.585	0.520	1.12	3
	11	3.94	5.24	0.67		0.244	4.6	1.20	2780	96.0	39.0	0.530	0.498	1.06	3
	15	3.94	5.24	0.83		0.244	4.6	1.18	3020	96.0	39.0	0.493	0.478	1.03	3
	4	3.94	5.24	0.55		0.528	4.6	2.56	2900	96.0	41.8	0.637	0.663	0.96	3
	12	3.94	5.24	0.51		0.528	4.6	2.56	2790	96.0	43.0	0.712	0.693	1.03	3
	16	4.02	5.32	0.87		0.528	4.6	2.47	2810	96.0	42.5	0.590	0.650	0.91	3

*(1) Both tension and compression reinforcement yielded before ultimate strength; (2) Only tension reinforcement yielded; (3) Only compression reinforcement yielded.

† Test failure moment in this series is taken as maximum moment before strain hardening of tension reinforcement commenced.

‡ Test failure moment in this series is taken as the moment recorded at which the concrete in the compression zone crushed.

In both the above series a further small increase in moment was obtained in several cases due to strain hardening of the tension reinforcement. However, this increase in load was accompanied by extremely large deflections and therefore was neglected. It should be noted that strain hardening of the structural or intermediate grade steel reinforcement only occurred in cases where either a very small percentage of tension reinforcement was provided or where an excessive amount of compression reinforcement was provided such that the compression reinforcement did not yield. For normal amounts of steel used in practice, the ultimate moment was not increased by strain hardening.

TABLE A-3—REINFORCED CONCRETE T-BEAMS REINFORCED IN TENSION ONLY

Source	Beam	b, in.	d, in.	b', in.	t, in.	A_s	p, per-cent	f_c', psi	f_y, ksi	$\frac{M}{bd^2 f_c'}$ Test	$\frac{M}{bd^2 f_c'}$ Calc	$\frac{M_{test}}{M_{calc}}$
A. N. Talbot[21]	1	16	10	8	3.25	1.68	1.05	1890	54.9	0.305	0.250	1.22
	2	32	10	8	3.25	3.36	1.05	1870	53.8	0.269	0.248	1.08
	3	24	10	8	3.25	2.24	0.93	1760	52.7	0.262	0.233	1.12
	4	16	10	8	3.25	1.76	1.10	1330	38.3	0.296	0.258	1.15
	5	32	10	8	3.25	3.36	1.05	1190	53.4	0.435	0.340	1.28
	6	24	10	8	3.25	2.20	0.92	1610	38.3	0.200	0.193	1.03
	7	16	10	8	3.25	1.76	1.10	1450	38.3	0.249	0.244	1.02
	8	24	10	8	3.25	2.20	0.92	1750	40.7	0.187	0.187	1.00
	9	32	10	8	3.25	3.08	0.97	1610	38.3	0.195	0.197	0.99
S. A. Guralnick[19]	IA-IR	23	11.81	7	4	2.08	0.766	3230	87.7	0.200	0.182	1.10
	IB-IR	23	11.81	7	4	1.20	0.442	2440	84.6	0.184	0.139	1.32
	IC-IR	23	11.78	7	4	3.72	1.375	4930	83.9	0.205	0.202	1.02
	ID-IR	23	11.81	7	4	2.08	0.766	4930	87.7	0.138	0.125	1.10
J. R. Gaston and E. Hognestad[22]	1	9	16.25	3.5	2.75 (avg)	1.20	0.82	4730	90.0	0.149	0.134	1.11
	2	9	16.00	3.5	2.75 (avg)	1.60	1.11	5230	90.0	0.185	0.159	1.16

Avg $M_{test}/M_{calc} = 1.11$. Standard deviation = 0.10.

TABLE A-4—PRESTRESSED CONCRETE BEAMS WITHOUT BOND

Source	Beam No.	b, in.	d, in.	p, per-cent	f_{se}, ksi	Steel tensile strength, ksi	f_c', psi	$\frac{M_{ult}}{bd^2 f_c'}$ Test	$\frac{M_{ult}}{bd^2 f_c'}$ Calc	$\frac{M_{test}}{M_{calc}}$
J. R. Janney, E. Hognestad, and D. McHenry[33]	3—0.128	6.0	8.3	0.322	110	235	5900	0.084	0.079	1.06
	3—0.144	6.0	8.3	0.322	122	235	5250	0.102	0.093	1.10
	3—0.307	6.0	8.3	0.644	130	235	4930	0.200	0.184	1.09
	3—0.428	6.0	8.3	0.965	120	235	5300	0.229	0.233	0.98
	3—0.428	6.0	8.3	0.965	133	235	5300	0.238	0.241	0.99
A. Feldman[24]	U-1	5.99	8.42	0.346	127.5	250.76	4240	0.139	0.135	1.03
	U-3	6.00	7.85	0.183	120.8	250.76	5620	0.055	0.048	1.15
	U-4	6.16	8.33	0.782	118.9	250.76	4060	0.274	0.250	1.10
	U-5	6.09	8.24	0.685	120.7	250.76	4060	0.273	0.271	1.01
	U-7	6.00	7.50	0.382	121.9	250.76	5020	0.120	0.124	0.97
	U-8	6.00	7.65	0.624	118.9	250.76	2565	0.285	0.290	0.95
	U-9	6.00	7.35	0.650	120.0	250.76	3550	0.273	0.246	1.11
	U-10	6.02	7.55	0.820	118.8	250.76	3390	0.328	0.292	1.12
	U-11	5.90	8.20	0.355	122.8	250.76	5490	0.121	0.107	1.13
	U-12	6.00	8.34	0.401	125.8	250.76	4020	0.173	0.145	1.19
	U-14	5.95	7.84	0.184	122.1	250.76	7600	0.049	0.043	1.14
	U-16	5.96	7.56	0.509	121.1	250.76	2190	0.308	0.287	1.07
R. J. Allen[25]	U-17	6.20	7.39	0.527	118.8	255	2120	0.330	0.304	1.09
	U-18	6.10	7.45	0.797	110.5	255	2770	0.339	0.321	1.06
	U-19	6.04	6.85	0.365	123.7	255	6270	0.113	0.101	1.12
	U-21	6.20	7.60	0.384	119.8	255	2450	0.218	0.223	0.98
	U-22	6.10	7.32	0.743	118.2	255	4910	0.227	0.215	1.05
	U-23	6.00	6.99	0.288	117.7	255	7580	0.049	0.067	0.72
	U-24	6.00	7.85	0.192	119.0	255	5660	0.064	0.061	1.05

Avg M_{test}/M_{calc} for 24 beams = 1.05. Standard deviation = 0.094.

TABLE A-5—PRESTRESSED CONCRETE BEAMS WITH WELL BONDED TENDONS

Source	Beam No.	b, in.	d, in.	p, per-cent	f_{se}, ksi	Steel tensile strength, ksi	f_c', psi	$\frac{M_{ult}}{bd^2 f_c'}$ Test	$\frac{M_{ult}}{bd^2 f_c'}$ Calc	$\frac{M_{test}}{M_{calc}}$
D. F. Billet and H. H. Appleton[12]	B2	6.15	9.53	0.198	116.8	246	5420	0.087	0.079	1.10
	B3	6.00	9.62	0.101	120.0	246	3750	0.065	0.061	1.06
	B7	6.13	8.09	0.942	112.8	248	5910	0.272	0.256	1.06
	B8	6.13	7.99	0.953	112.9	248	3280	0.462	0.349	1.32
	B9	6.06	9.23	0.418	19.1	240	6330	0.129	0.128	1.01
	B10	6.06	9.01	0.107	19.0	240	3530	0.068	0.068	1.00
	B11	6.06	9.21	0.419	20.4	240	3910	0.208	0.190	1.09
	B13	6.02	8.15	0.656	21.2	240	3750	0.283	0.242	1.17
	B14	6.00	7.99	0.916	20.2	240	3750	0.327	0.266	1.23
	B15	6.03	9.29	0.418	150.0	240	5710	0.143	0.143	1.00
	B16	6.01	9.00	0.108	150.3	240	3330	0.077	0.073	1.05
	B17	6.00	9.09	0.429	151.0	240	4580	0.179	0.178	1.01
	B18	6.00	8.29	0.647	148.8	240	4100	0.273	0.264	1.03
	B19	6.08	8.27	0.873	151.3	240	6220	0.244	0.242	1.01
	B21	6.08	9.05	0.284	118.0	248	6560	0.093	0.098	0.95
	B22	6.07	9.13	0.561	115.2	248	7630	0.153	0.150	1.02
	B23	6.04	8.20	0.943	117.3	248	8200	0.213	0.206	1.05
	B24	6.07	8.24	0.746	116.4	248	6120	0.235	0.221	1.06
	B25	6.06	8.01	0.641	114.5	248	3270	0.349	0.294	1.19
	B27	6.07	8.36	0.920	118.0	248	4590	0.319	0.295	1.08
A. Feldman[24]	B28	6.15	7.93	0.475	92.5	186	2500	0.281	0.260	1.08
	B29	6.16	8.07	0.815	92.7	186	4280	0.261	0.240	1.09
	B30	6.09	8.08	0.177	101.1	248	2890	0.135	0.129	1.05
	B31	6.08	8.23	0.579	94.1	248	3450	0.279	0.250	1.12
	B32	6.00	9.32	0.510	115.3	256	7180	0.154	0.158	0.97
	B33	6.03	9.08	0.312	116.9	256	8320	0.093	0.091	1.02
J. R. Janney, E. Hognestad, and D. McHenry[23]	1—0.141	6	8.3	0.322	119	235	5350	0.144	0.127	1.13
	1—0.250	6	8.3	0.644	113	235	6050	0.218	0.186	1.17
	1—0.420	6	8.3	0.965	117	235	5400	0.260	0.258	1.01
	2—0.151	6	8.3	0.322	126	235	5000	0.146	0.134	1.07
	2—0.306	6	8.3	0.644	118	235	4950	0.237	0.214	1.11
	2—0.398	6	8.3	0.965	117	235	5700	0.271	0.251	1.08

Average M_{test}/M_{calc}: All 32 beams = 1.07. Standard deviation = 0.077.

TABLE A-6—UNSYMMETRIC REINFORCED CONCRETE T-BEAMS

Source	Beam	Flange width, B, in.	Effective depth, d, in.	f_c', psi	A_s, sq in.	f_y, ksi	Ultimate moment, in.-kip		$\frac{M_{test}}{M_{calc}}$	Type of failure
							Calc	Test		
A. H. Mattock and L. B. Kriz[28]	L1	8	13.8	2820	1.20	45.6	647	678	1.05	T
	L2	8	13.8	6200	1.20	46.1	710	693	0.98	T
	L3	8	12.9	3260	2.40	45.6	952	1062	1.12	C
	L4	8	12.9	5570	2.40	51.0	1300	1325	1.02	CT
	L5	12	13.8	2050	1.20	47.9	627	671	1.07	T
	L6	12	13.8	6320	1.20	47.5	733	778	1.06	T
	L7	12	12.9	2225	2.40	43.0	695	876	1.26	C
	L8	12	12.9	6420	2.40	43.0	1160	1173	1.01	T
	L9	16	13.8	2680	1.20	45.4	637	663	1.04	T
	L10	16	13.8	4820	1.20	44.5	678	757	1.12	T
	L11	16	12.9	2530	2.40	50.7	775	1129	1.46	C
	L12	16	12.9	5820	2.40	50.5	1298	1265	0.97	T

(a) All beams: Avg $M_{test}/M_{calc} = 1.10$; Standard deviation = 0.138.
(b) Beams with ultimate strength controlled by tension: Avg $M_{test}/M_{calc} = 1.04$; Standard deviation = 0.047.

TABLE A-7—REINFORCED CONCRETE BEAMS WITH TRIANGULAR COMPRESSION ZONES

Source	Beam	f_y, psi	f_y', ksi	A_s, sq in.	Ultimate moment, in.-kip		$\frac{M_{test}}{M_{calc}}$
					Calc	Test	
A. H. Mattock and L. B. Kriz[28]	T1	3620	49.0	0.88	194.5	196.8	1.01
	T2	3455	49.0	1.32	215.0	240.9	1.12
	T3	6290	45.0	1.80	320.0	326.8	1.02
	T4	1735	49.5	0.88	116.3	131.8	1.13
	T5	3500	50.5	1.32	219.0	258.4	1.18
	T6	7000	45.0	1.80	323.0	368.3	1.14

For all beams, $d = 7$ in., $\theta = 45$ deg.
Ultimate strength of all beams controlled by crushing of concrete.
Avg $M_{test}/M_{calc} = 1.10$. Standard deviation = 0.069.

TABLE A-8—TIED REINFORCED CONCRETE COLUMNS LOADED CONCENTRICALLY*

Column	Percentage reinforcement	Yield point of reinforcement, psi	f_c', psi	Ultimate load, kip		$\frac{P_{test}}{P_{calc}}$
				Test	Calc	
a	4	50,000	2860	219	231	0.95
b	4	50,000	3090	255	242	1.05
a	4	50,000	2650	253	222	1.14
b	4	50,000	2850	238	231	1.03
a	1.5	44,700	4700	225	246	0.92
b	1.5	44,700	4150	227	222	1.02
a	4	50,000	4670	285	310	0.92
b	4	50,000	4730	320	313	1.02
a	4	50,000	4225	293	291	1.01
b	4	50,000	4570	309	306	1.01

Continued on p. 926

TABLE A-8 (cont.)—TIED REINFORCED CONCRETE COLUMNS LOADED ECCENTRICALLY*

Column	Percentage reinforcement	Yield point of reinforcement, psi	f_c', psi	Ultimate load, kip		P_{test}/P_{calc}
				Test	Calc	
a	6	42,200	4215	317	315	1.01
b	6	42,200	4985	291	348	0.84
a	4	50,000	5870	353	363	0.97
b	4	50,000	6950	387	410	0.94
a	4	50,000	6245	410	379	1.08
b	4	50,000	6530	420	391	1.08

Average value of $P_{test}/P_{calc} = 1.00$. Standard deviation $= 0.074$.
*Source: Series 3 of ACI column investigation—University of Illinois *Bulletin* No. 267.

TABLE A-9—RECTANGULAR REINFORCED CONCRETE COLUMNS LOADED ECCENTRICALLY

Source and characteristics	Column No.	Concrete cylinder strength, f_c', psi	Eccentricity (total), in.	Ultimate load, kip		P_{test}/P_{calc}	Mode of failure
				Test	Calc		
Group I							
E. Hognestad[8]	A-1a	5280	0.12	388	452	0.86	C
	A-1b	5660	0.14	441	481	0.92	C
$b = t = 10$ in.	B-1a	4250	0.12	343	372	0.92	C
	B-1b	4070	0.12	352	356	0.99	C
$A_s = 1.24$ sq in.	C-1a	2270	0.13	222	212	1.05	C
	C-1b	2020	0.13	191	192	1.00	C
$A_s' = 0.22$ sq in.	A-2a	5280	2.72	239	240	1.00	C
	A-2b	5830	2.78	253	260	0.97	C
$f_y = 43.6$ ksi	B-2a	4250	2.77	213	206	1.03	C
	B-2b	4070	2.74	190	197	0.96	C
$E_s = 28 \times 10^6$ psi	C-2a	2270	2.77	118.5	116	1.02	C
	C-2b	1970	2.77	100.0	103	0.97	C
$f_y' = 60.0$ ksi	A-3a	5660	5.32	133.5	154.0	0.87	T
	A-3b	5830	5.28	140.0	158.0	0.89	T
$d = 8.67$ in.	B-3a	4630	5.41	125.9	134.3	0.94	CT
	B-3b	4290	5.37	116.0	129.4	0.90	CT
$d' = 1.33$ in.	C-3a	1880	5.28	60.5	66	0.92	C
	C-3b	1690	5.33	64.0	52	1.03	C
	A-4a	4810	7.95	84.5	85.7	0.99	T
	A-4b	5600	7.85	81.0	92.1	0.88	T
	B-4a	3800	7.98	80.0	77.7	1.03	T
	B-4b	4290	8.02	81.0	81.1	1.00	T
	C-4a	1690	7.82	50.5	48	1.05	C
	C-4b	1730	7.81	52.0	49	1.06	C
	A-5a	4810	12.90	48.2	45.6	1.06	T
	A-5b	5600	12.90	42.8	46.9	0.91	T
	B-5a	4290	12.92	46.1	44.5	1.04	T
	B-5b	4590	12.95	45.5	45.0	1.01	T
	C-5a	2310	12.84	39.0	37.7	1.04	CT
	C-5b	1770	12.84	32.8	34.0	0.96	CT
Group II							
E. Hognestad[8]	B-6a	4080	0.07	456	437	1.04	C
	B-6b	4040	0.06	420	436	0.96	C
$b = t = 10$ in.	C-6a	2020	0.10	225	268	0.84	C
	C-6b	1520	0.18	202	222	0.91	C

TABLE A-9 (cont.)—RECTANGULAR REINFORCED CONCRETE COLUMNS LOADED ECCENTRICALLY

Source and characteristics	Column No.	Concrete cylinder strength, f_c', psi	Eccentricity (total), in.	Ultimate load, kip		P_{test}/P_{calc}	Mode of failure
				Test	Calc		
$A_s = 1.24$ sq in.	A-7a	5240	3.44	274	254	1.08	C
	A-7b	5810	2.76	284	312	0.91	C
$A_s' = 1.24$ sq in.	B-7a	4080	2.75	256	245	1.04	C
	B-7b	4040	2.74	248	244	1.02	C
$f_y = 43.6$ psi	C-7a	1970	2.78	141	147	0.96	C
	C-7b	1520	2.77	126.8	135	0.94	C
$E_s = 28 \times 10^6$ psi	A-8a	5520	5.34	162	177.9	0.91	T
	A-8b	5810	5.40	152	179.5	0.85	T
$f_y = 43.6$ psi	B-8a	4700	5.35	156	164.9	0.95	CT
	B-8b	4260	5.32	146	158.6	0.92	CT
$d = 8.67$ in.	C-8a	1820	5.32	99	103	0.96	C
	C-8b	1820	5.39	99	102	0.97	C
$d' = 1.33$ in.	A-9a	5100	7.87	89.0	99.1	0.90	T
	A-9b	5170	7.89	91.2	99.1	0.92	T
	B-9a	4700	7.85	94.0	97.7	0.96	T
	B-9b	4370	7.82	89.5	96.6	0.93	T
	C-9a	1880	7.88	73.0	75.8	0.96	CT
	C-9b	1730	7.85	65.5	74.1	0.88	CT
	A-10a	5100	12.78	46.1	47.8	0.97	T
	A-10b	5170	12.75	44.0	48.0	0.92	T
	B-10a	4260	12.78	43.5	47.2	0.92	T
	B-10b	4370	12.79	44.0	47.2	0.93	T
	C-10a	2300	12.85	44.5	44.3	1.00	T
	C-10b	1770	12.88	45.0	42.6	1.06	T
Group III							
E. Hognestad[8]	B-11a	3870	0.08	500	513	0.98	C
	B-11b	4070	0.10	485	517	0.94	C
$b = t = 10$ in.	C-11b	2070	0.00	353	376	0.94	C
	A-12a	4150	2.70	315	306	1.03	C
$A_s = 2.40$ sq in.	A-12b	5050	2.72	326	343	0.95	C
	B-12a	4300	2.72	303	318	0.95	C
$A_s' = 2.40$ sq in.	B-12b	4010	2.76	284	298	0.95	C
	C-12a	2300	2.76	252	224	1.12	C
$f_y = 43.6$ ksi	C-12b	2200	2.72	230	218	1.05	C
	A-13a	5350	5.36	220	227	0.97	C
$E_s = 29 \times 10^6$ psi	A-13b	4850	5.34	210	216	0.97	C
	B-13a	3580	5.35	180	188	0.96	C
$f_y = 43.6$ ksi	B-13b	4290	5.34	206	206	1.00	C
	C-13a	2300	5.33	151	153	0.99	C
$d = 8.50$ in.	C-13b	2070	5.28	137	148	0.93	C
	A-14a	5350	7.87	142	159.0	0.89	T
$d' = 1.50$ in.	A-14b	5100	7.93	153	155.5	0.98	T
	B-14a	3580	7.89	138.8	141.0	0.98	CT
	C-14a	1950	7.84	115.5	108	1.07	C
	C-14b	2070	7.87	104.0	111	0.94	C
	A-15a	5100	12.92	88.0	82.8	1.06	T
	A-15b	4850	12.85	79.0	83.0	0.95	T
	B-15a	3800	12.91	74.0	80.3	0.92	T
	B-15b	4630	12.92	84.5	82.0	1.03	T
	C-15a	1950	12.89	72.5	74	0.97	C
	C-15b	2070	12.91	74.5	73.5	1.01	CT

Avg $P_{test}/P_{calc} = 0.97$. Standard deviation $= 0.059$.
Note: The eccentricity recorded in this table is measured from the center of the column.

TABLE A-10—CIRCULAR SECTION COLUMNS SUBJECT TO ECCENTRIC LOAD*

Column No.	f_c', psi	Eccentricity, e, in.	Ultimate load, kips		P_{test}/P_{calc}	Cross section
			Test	Calc		
A-16a	5150	0	760	704	1.08	All columns 12-in. diameter.
A-16b	4640	0.01	693	655	1.06	
B-16a	2990	0.03	515	497	1.03	Reinforcement: eight ⅞-in. diameter
B-16b	3310	0.01	514	527	0.98	
C-16a	1590	0.03	371	362	1.02	
C-16b	1420	0.02	365	345	1.06	f_y = 43.8 ksi
A-17a	5150	3.30	343	308.6	1.11	
A-17b	4640	3.29	283	291.9	0.97	
B-17a	3620	3.34	253	251.7	1.01	
B-17b	3310	3.34	238	239.3	0.99	
C-17a	1420	3.55	187	160.0	1.17	
C-17b	1600	3.50	179	164.9	1.08	
A-18a	5020	6.44	162	164.3	0.99	
A-18b	5000	6.50	171	162.5	1.05	
B-18a	3380	6.42	140	140.8	0.99	
B-18b	3580	6.47	136	143	0.95	
C-18a	1680	6.80	127	100.6	1.26	
C-18b	1590	6.60	107	101.9	1.05	
A-19a	5020	9.62	111.0	107.2	1.04	
A-19b	5310	9.62	114.3	109.4	1.04	
B-19a	3380	9.54	98.5	93.2	1.06	
B-19b	3580	9.56	103.0	95.4	1.08	
C-19a	1680	9.80	79.0	72.9	1.08	
C-19b	1630	9.80	79.0	72.4	1.09	
A-20a	5310	15.68	67.7	63.0	1.07	
A-20b	5000	15.58	63.5	62.0	1.02	
B-20a	2990	15.75	57.5	53.0	1.08	
B-20b	3620	15.60	62.0	57.0	1.09	
C-20a	1630	15.60	47.0	45.8	1.03	
C-20b	1600	15.72	47.0	44.9	1.05	

Avg P_{test}/P_{calc} for all 30 columns = 1.05
Standard deviation = 0.060
*Source: E. Hognestad.[8]

TABLE A-11 — RECTANGULAR COLUMNS SUBJECT TO AXIAL THRUST AND BENDING ABOUT TWO AXES*

Column No.	b, in.	r, in.	d', in.	Reinforcement	f_y, psi	f_c', psi	e_x', in.	e_y', in.	P_{test}, kip	P_{calc}, kip	P_{test}/P_{calc}
SC1	4	4	0.75	4-1/4 diameter	35.8	5435	3.14	3.14	5.27	5.10	1.03
SC6									4.68		0.92
SC2	4	4	0.75	4-5/16 diameter	39.2	5435	3.07	3.07	7.44	7.46	1.00
SC7									7.25		0.97
SC3	4	4	0.75	4-3/8 diameter	40.2	5435	3.06	3.05	9.50	9.13	1.04
SC8									9.81		1.07
SC4	4	4	0.75	4-1/2 diameter	45.7	5435	3.03	3.03	13.50	13.88	0.97
SC9									14.30		1.03
SC5	4	4	0.75	4-5/8 diameter	40.7	5435	3.02	3.02	16.48	17.34	0.95
SC10									16.50		0.95

Avg P_{test}/P_{calc} = 0.99 Standard deviation = 0.046
*Source: P. Anderson and H. N. Lee.[27]