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REDISTRIBUTION OF DESIGN BENDING MOMENTS IN REINFORCED CONCRETE CONTINUOUS BEAMS

by

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For written discussion

SYNOPSIS

Arbitrary redistribution of design bending moments in continuous reinforced concrete beams is discussed.

Tests of two series of continuous reinforced concrete beams are described. Series 1 was a pilot series of four small two-span continuous beams designed for the same working load, but using various distributions of design bending moments. Series 2 consisted of three continuous beams intended to simulate secondary beams in a reinforced concrete frame building. One beam, reinforced with mild steel was designed for the distribution of bending moments predicted by the elastic theory. Two beams were designed using a 25% reduction of support-section moment with appropriate increase in span-section design moments. Of these two beams one was reinforced with mild steel and the other with work-hardened square twisted steel.

The test results are discussed, and it is concluded that redistribution of design bending moments by up to 25% does not result in performance inferior to that of beams designed for the distribution of bending moments predicted by the elastic theory, either at working loads or at failure.

INTRODUCTION

REDISTRIBUTION of design bending moments in continuous reinforced concrete beams is widely recognized as a most useful tool in the hands of the designer of reinforced concrete structures. The arbitrary reduction of bending moments at supports, initially calculated using the elastic theory, leads to a reduction in congestion of reinforcement at support sections. This in turn makes better compaction of the concrete possible and enables detailing of reinforcement to be simplified. For beams in which the live load/permanent load ratio is high, and in which the live load can be applied to the spans in several different ways, redistribution of the design bending moments can result in a reduction of the maximum design bending moments both in the spans and at the supports. This of course leads to smaller sections throughout the beam. Used with discretion, arbitrary redistribution of design bending moments in continuous reinforced concrete beams can therefore result in sounder and more economic structures.

2. The justification for arbitrary redistribution of design bending moments lies in the elasto-plastic behaviour of reinforced concrete sections, which has been demonstrated on many occasions [1, 2, 3, 4, 5, 6]. This behaviour ensures that the redistribution of moments assumed in design will actually occur before failure of the beam. It can easily be demonstrated that for beams failing by yield of the steel, as is the case with beam sections designed according to B.S.C.P. 114 (1957), the factor of safety against collapse of a beam is not affected by arbitrary redistribution of design bending moments.

3. Limitations on the amount of arbitrary redistribution of design moments allowed under the current B.S.C.P. 114 (1957), arise from consideration of the performance of beams in the working load range rather than at failure. In order that structures may be serviceable, large deflexions and excessive cracking must be avoided at working load. Unrestricted redistribution of design moments could lead to over-high stresses in the reinforcement at certain sections under working load conditions, and this in turn could possibly result in excessive deflexion and cracking.

SCOPE OF INVESTIGATION

4. The object of previous work has been to show that any assumed redistribution of bending moments would, in fact, occur before failure, and so ensure an adequate factor of safety. This investigation is concerned primarily with the influence of arbitrary redistribution of design bending moments on the performance of continuous reinforced concrete beams at design load. The tests were also carried through to failure of the beams, in order to accumulate additional evidence of the elasto-plastic behaviour of reinforced concrete.

TEST PROGRAMME

5. Two series of continuous reinforced concrete beams were tested to destruction in flexure.

(*a*) *Series 1*

6. This was a pilot series of four rectangular two-span continuous beams of overall dimensions 13 ft × 9 in. × 4 in. The loading scheme used is shown diagramatically in Fig. 1, together with the distribution of bending moments given by the elastic theory assuming constant stiffness of the beam section.

7. The first beam was designed for the bending-moment distribution given by the elastic theory, and each succeeding beam had its design moments redistributed by a progressively larger amount.

8. These beams were designed according to the "straight-line" theory (based on design load), maximum stresses of 1,500 lb/sq. in. for concrete and 20,000 lb/sq. in. for steel. Assuming a mean effective depth of 7·5 in., these stresses lead to a maximum design resisting moment of 74,300 lb.-in., the required area of reinforcement being 0·60 sq. in. Actually, twelve ¼-in.-dia. bars were used, giving a cross-section of 0·59 sq. in. The bending moment of 74,300 lb.-in. was taken as the design moment for section B of beam No. 1. This moment would be produced by a design load $P = 2{\cdot}25$ tons

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[1] The references are given on p. 46.

plus the self-weight of the beam. The support section C of beam No. 1 was designed for the "elastic" bending moment corresponding to this design load.

9. In beams Nos 2, 3, and 4, the cross-sectional area of steel at section B was progressively reduced, and the area of steel at section C was increased so that in every case:

$$M_B{}^d + \frac{M_C{}^d}{2} = \frac{P.L}{4}$$

where $M_B{}^d$ and $M_C{}^d$ are the design bending moments for sections B and C.

10. The design moments and reinforcement details are listed in Table 1.

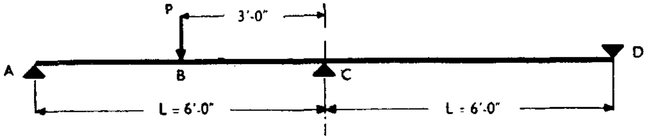
TABLE 1

Beam No.	Design moments: (pounds-inches) M_B	M_C	Percentage redistribution (Percentage reduction in $M_B{}^d$	Section B		Section C	
				Number of ¼-in.-dia. bars	Effective depth: (inches)	Number of ¼-in.-dia. bars	Effective depth: (inches)
1	74,300	34,400	0	12	7·5	6	7·5
2	67,300	48,400	9·4	11	7·5	8	7·5
3	61,500	60,000	17·2	10	7·5	10	7·5
4	51,700	79,600	30·4	8	7·75	13	7·5

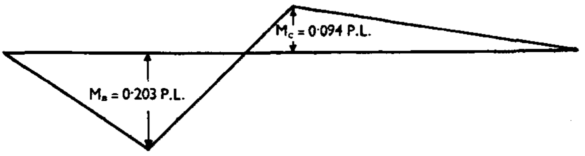
The details of a typical beam of this series are shown in Fig. 2. The concrete was made with ⅜ in. maximum-size aggregate, and was designed to yield a cube compressive strength of 4,500 lb/sq. in. at 28 days.

(b) Series 2

11. The three beams used were intended to simulate secondary beams in a reinforced concrete frame building. These beams each had two equal spans of 15 ft and the design load on each span was 4 tons. The loading scheme and the distribution of bending moments given by the elastic theory, assuming



(a) TEST LOADING SCHEME, SERIES I



(b) DISTRIBUTION OF BENDING MOMENT BY ELASTIC THEORY, ASSUMING BEAM-SECTION STIFFNESS TO BE CONSTANT

FIG. 1.—SERIES 1 TESTS

constant stiffness along the length of the beam, are shown in Fig. 3 (a and b). Beam NR 1 was designed for the "elastic" distribution of bending moments. In beams R 1 and R 2 the design bending moment at the centre supports was made 25% less than the calculated "elastic" moment, and the design moments in the span were increased accordingly. This distribution is shown in Fig. 3c. Beams NR 1 and R 1 were designed assuming the use of smooth round mild-steel bars, minimum yield point 40,000 lb/sq. in., and beam R 2 assuming the use of square twisted work-hardened reinforcement, minimum yield point 60,000 lb/sq. in. All the beams were designed as singly reinforced T-sections in the spans and doubly reinforced rectangular sections at the centre supports. Concrete with a cube crushing strength of 3,000 lb/sq. in. was assumed in the design of all three

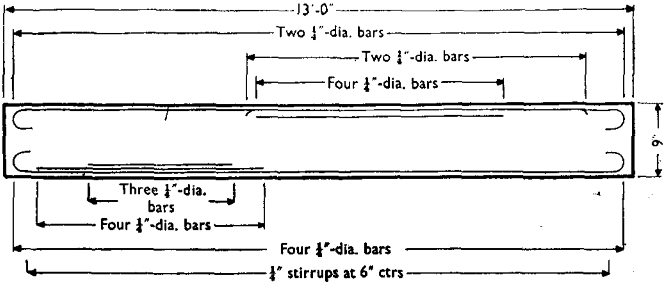
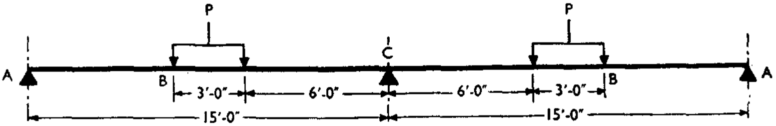
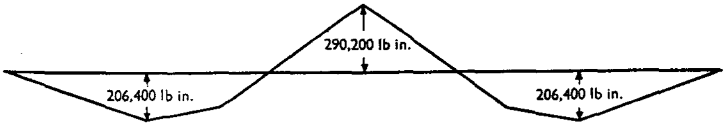


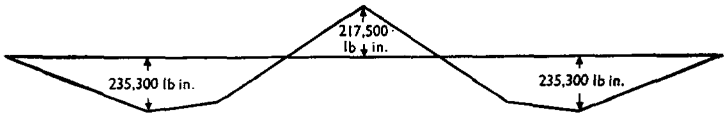
FIG. 2.—DETAILS OF TYPICAL BEAM, SERIES 1
(Beam No. 2. Vertical scale twice horizontal scale)



(a) TEST LOADING SCHEME, SERIES 2



(b) DISTRIBUTION OF BENDING MOMENT BY ELASTIC THEORY FOR P = 4 TONS (STIFFNESS OF BEAM ASSUMED CONSTANT ALONG ITS ENTIRE LENGTH)



(c) DISTRIBUTION OF BENDING MOMENTS FOR P = 4 TONS AFTER 25% REDISTRIBUTION

FIG. 3.—SERIES 2 TESTS

beams. The individual sections were designed by the load-factor method contained in clause 306 of B.S.C.P. 114 (1957). Details of the beams are shown in Fig. 4. The width of the T-flange was determined by the available clearance in the test frame.

TEST PROCEDURE

(*a*) *Series 1*

12. The beams were loaded to destruction by increments of ½ ton or 1 ton, and at each load stage the following measurements were made:—

(i) Load on the beam using a proving ring interposed between the hydraulic jack and the loading plate.

(ii) Centre-support reaction reading direct from the 10-ton-capacity Macklow-Smith pressure capsule.

(iii) Deflexion of beam under the loading point relative to supports A and C using independently mounted dial gauges.

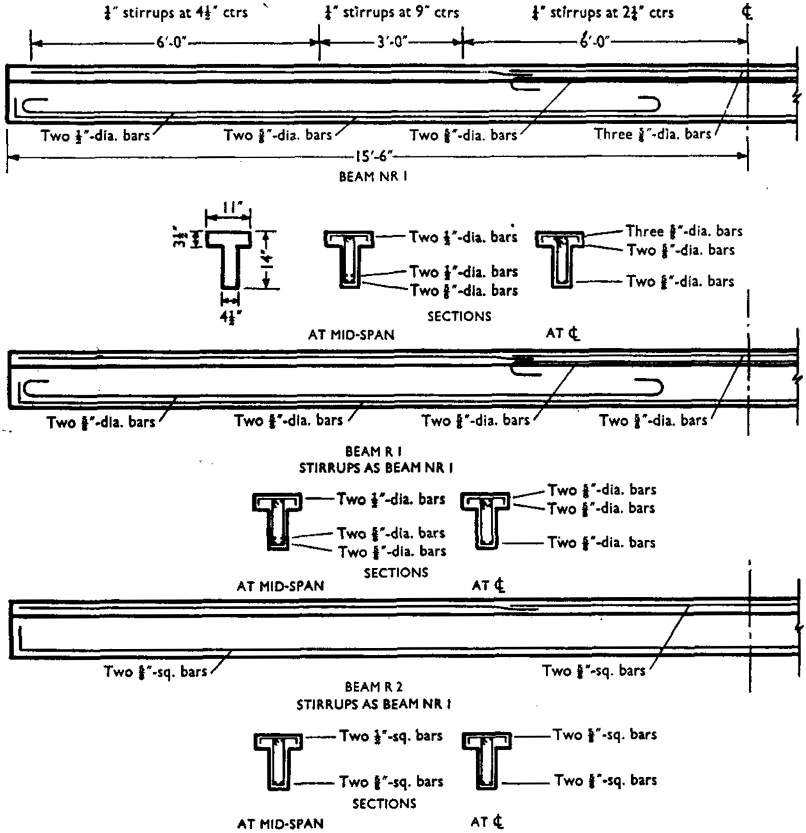
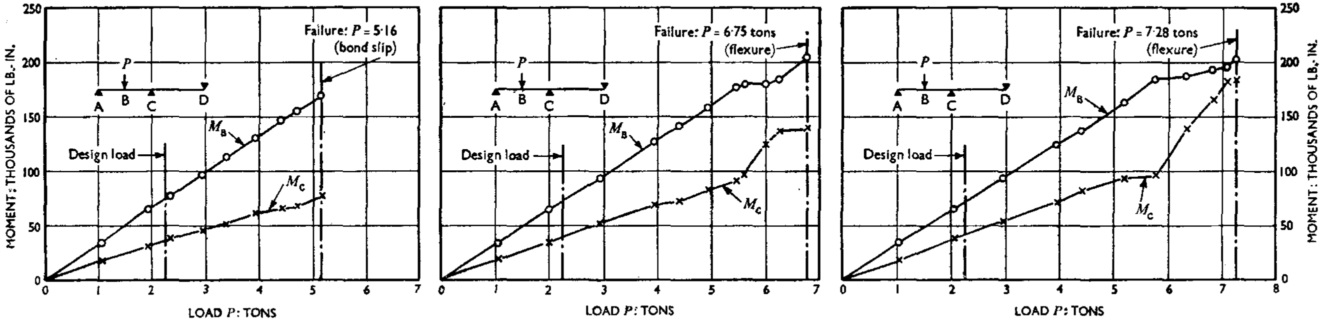


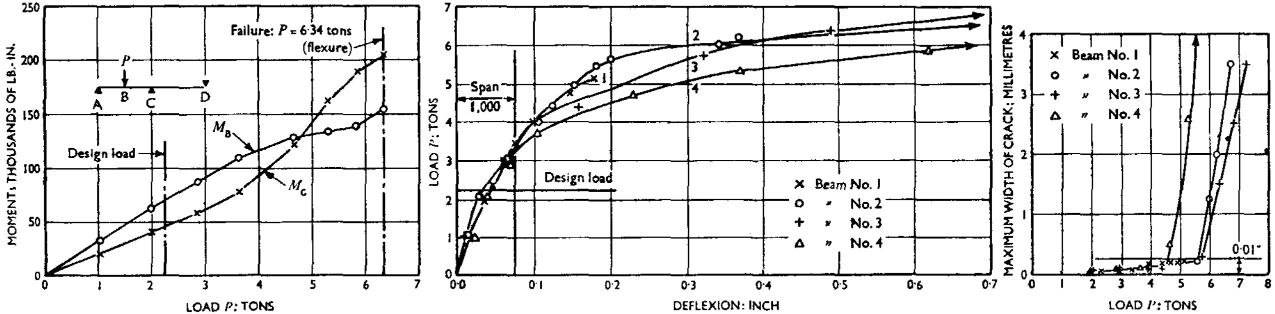
FIG. 4.—DETAILS OF BEAMS, SERIES 2



(a) MEASURED MOMENTS PLOTTED AGAINST LOAD: BEAM NO. 1

(b) MEASURED MOMENTS PLOTTED AGAINST LOAD: BEAM NO. 2

(c) MEASURED MOMENTS PLOTTED AGAINST LOAD: BEAM NO. 3



(d) MEASURED MOMENTS PLOTTED AGAINST LOAD: BEAM NO. 4

(e) LOAD PLOTTED AGAINST DEFLEXION: ALL BEAMS, SERIES 1

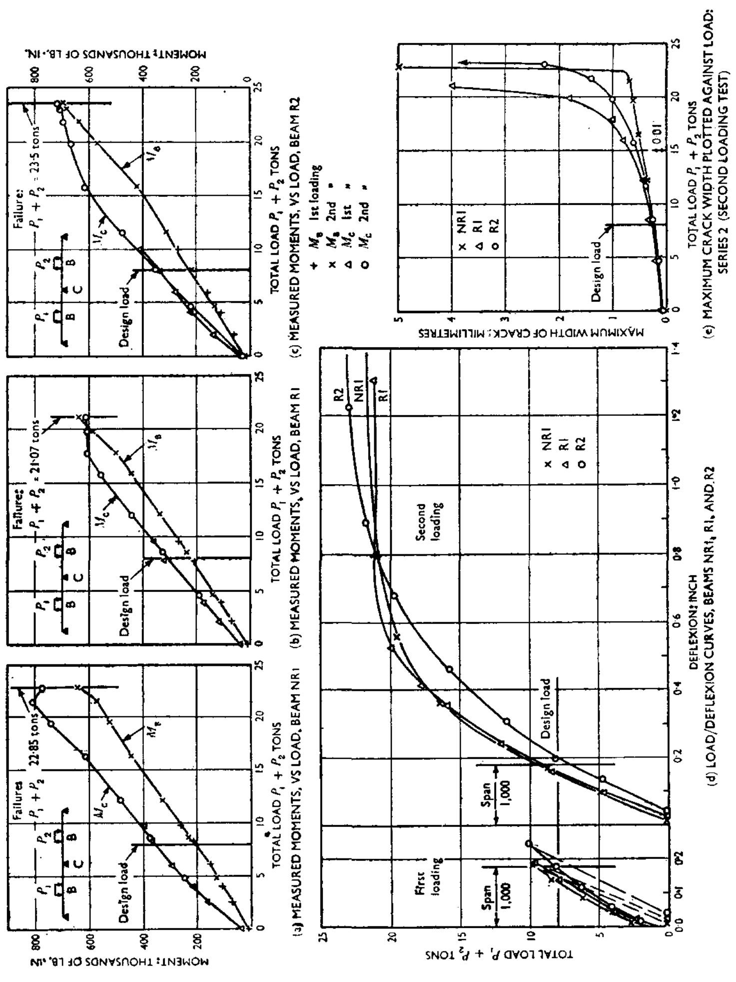
(f) MAXIMUM CRACK WIDTH PLOTTED AGAINST LOAD: ALL BEAMS, SERIES 1

FIG. 5.—TEST RESULTS, SERIES 1

(iv) Strains across the depth of the beam at sections B and C, using an 8-in-gauge-length demountable mechanical strain gauge.

(v) Maximum width of crack at the level of the tension reinforcement. This was measured using a portable hand microscope having a scale graduated in tenths of millimetres. Crack widths were estimated to ¼ of a division.

13. After each increment of load the levels of the three supports were measured and, if necessary, the level of support C was adjusted to make the three supports co-linear. The measurements (i) to (iii) above were made after this leveling procedure had been completed. The maximum adjustment to the



(a) MEASURED MOMENTS, VS LOAD, BEAM NR1

(b) MEASURED MOMENTS, VS LOAD, BEAM R1

(c) MEASURED MOMENTS, VS LOAD, BEAM R2

(d) LOAD/DEFLEXION CURVES, BEAMS NR1, R1, AND R2

(e) MAXIMUM CRACK WIDTH PLOTTED AGAINST LOAD: SERIES 2 (SECOND LOADING TEST)

FIG. 6.—TEST RESULTS, SERIES 2

level of support C necessary at any load stage was 0·009 in., and the average adjustment was 0·005 in.

(*b*) *Series 2*

14. Two loading tests were carried out on each beam. In the first test the load was increased by increments of approximately 1 ton up to 1¼ times the design load, i.e. a total of 5 tons on each span. The load was removed and a second test was carried through to destruction of the beam. In this test the load was increased by increments of approximately 2 tons until failure was approached when the increments were reduced. No rest period was allowed between the two load tests.

15. At each load stage the following measurements were made:—

(i) Load on the beam measured by proving rings mounted between the hydraulic jacks and the load-distribution beams in each span.
(ii) Centre-support reaction, reading direct from a 50-ton capacity Macklow-Smith load capsule.
(iii) Level of the three supports using cathetometers sighting on targets attached to the beam.
(iv) Deflexion of beam under outer load point relative to supports using cathetometers as in (iii).
(v) Maximum width of crack at level of tension reinforcement. As in series 1 this was measured using a hand microscope.

16. After application of each increment of load the level of the centre support was adjusted to keep the supports co-linear. The measurements of load, centre-support reaction, deflexion and maximum crack width were made after levelling of supports. After a settlement of centre support of 0·03 in. at first load increment, the average adjustment in level of this support was 0·002 in.

TEST RESULTS

17. For the sake of brevity the results are set out in graphical form in Figs 5 and 6. The following measurements are plotted against the total load acting on the beam:—

(i) Bending moment at point of application of load and at centre support,
(ii) Deflexion under load point,
(iii) Maximum width of crack at level of main tension steel.

18. In the case of series 2 beams the plotted bending moment at point of application of outer load and plotted deflexion under this load point are the average of the values in both spans. The properties of the materials used are as shown in Table 2.

DISCUSSION OF TEST RESULTS

(*a*) *In the design-load range*

19. *Test Series 1.* Inspection of the plots of moment against load, in the working load range, shows that redistribution of moments was already taking place, even though the steel stresses were well below the yield-point stress. The actual bending moments at section B, at "working load" are shown in Table 3, and may be compared with the working load bending moment, calculated by elastic theory, of 74,300 lb.-in.

20. It is seen that there is an actual redistribution of bending moments at working load amounting to slightly more than one quarter of the arbitrary amount of redistribution of bending moments assumed in design. This redistribution occurs because the moment/rotation relation for a reinforced concrete section is not a straight line for low loads, as is assumed in the elastic analysis, but is in fact slightly curved. The moment/rotation curve for section B in Beam No. 3 is shown in Fig. 7. The rotations were calculated for a length of beam equal to its effective depth, using the strains measured at the section. It can be seen that the stiffness of the section decreases as the applied moment increases. This behaviour is typical of all reinforced concrete beam sections failing by yield of the steel. In a continuous beam, the design moments of which have been redistributed, sections for which the design moments have been reduced will be overstressed and will have a reduced stiffness. Conversely, understressed sections will be stiffer. These changes in stiffness automatically result in a redistribution of bending moments in the beam; bending moments will reduce in the region of reduced stiffness and increase in regions of increased

TABLE 2

Test Series	Beam No.	Concrete tube crushing strength at time of test: lb/sq. in.			Steel reinforcement		
					Type	Yield stress: lb/sq. in.	Ultimate stress: lb/sq. in.
1	1	4,900			¼-in.-dia. bars	57,500	76,200
	2	4,500					
	3	4,200					
	4	4,050					
		Section					
		B (North)	C	B (South)			
	NR 2	3,000	2,700	3,100	⅝-in.-dia. mild-steel bars	46,600	65,000
	R 1	3,050	3,000	2,950			
	R 2	3,100	3,300	2,850	⅝-in. sq. bars	71,000	80,300

TABLE 3

Beam No.	Measured M_B: lb.-in.	Actual redistribution (A): per cent	Design redistribution (D): per cent	$\frac{(A)}{(D)}$
1	74,300	0	0	—
2	72,500	2·42	9·4	0·257
3	71,000	4·44	17·2	0·258
4	67,000	9·83	30·4	0·232

stiffness. The partial redistribution of bending moments at design load is beneficial in that it leads to reduced crack widths and deflexions.

21. At working loads the cracking and deflexion of the beams for which the design moments had been redistributed, was no more severe than that of the beam designed for the elastic-theory distribution of moments. The maximum width of crack is almost identical for all beams of this series up to approximately twice the design load. The generally accepted limit to crack width at working load is 0·01 in. but in the beams of series 1 this figure was not exceeded until the main tensile reinforcement yielded. Since small-diameter bars were used for the main reinforcement only fine cracks were to be expected at the design stress for the reinforcement. More importance is attached therefore to the relative magnitude of the cracks in the beams tested, than to their absolute magnitude.

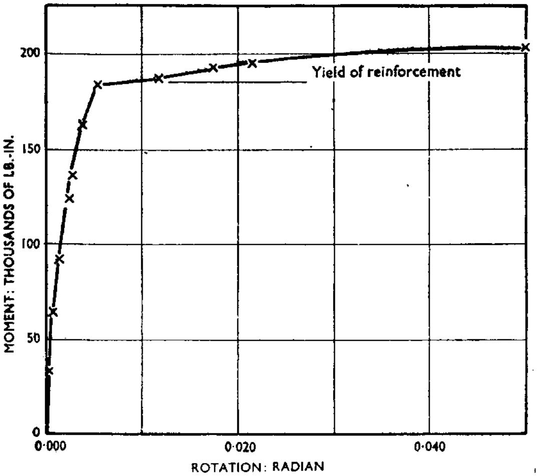


FIG. 7.—MOMENT/ROTATION CURVE, SECTION B, BEAM NO. 3

22. The deflexion of all the beams of series 1 was very nearly the same for loads up to 3 tons, i.e. approximately 1¼ times the working load.

23. *Test Series 2.* Examination of the moment/load curves for the beams of series 2, reveals that redistribution of moments was occurring at working load in the case of beam R 1, but not in the case of beam R 2. This may be because the compression reinforcement supplied over the centre support in beam R 2 was excessive, the two ⅝-in-sq. bars being carried from end to end of the beam for simplicity in detailing. Curvature of the initial part of the moment/rotation curve for a reinforced concrete section occurs because the stress/strain relation for concrete is a curve and not a straight line. A large amount of compression steel may mask the influence of the curvature of the concrete stress/strain relation on the moment/rotation curve for the section considered. The percentage redistribution of moment which had already occurred at working load in beam R 1 was 6·9%, representing 27·5% of the arbitrary redistribution

assumed in design. This is approximately the same degree of redistribution as occurred in the beams of series 1.

24. The deflexion results were even more favourable in this series than in series 1. For all ranges of load the deflexions of beams NR 1 and R 1 were almost identical. The extra deflexion of beam R 2 was to be expected, since the design steel stress was 50% higher in this beam than in beams NR 1 and R 1.

25. The immediate recovery of deflexion on unloading from 1¼ times design load was 92% for beam NR 1 and 85% for both beams R 1 and R 2.

26. The maximum width of crack was very nearly the same in all three beams of this series for loads up to 1½ times the design load. At design load the maximum width of crack was approximately equal to the generally accepted limit to crack width of 0·01 in. At 1½ times design load the maximum width of crack was still only 0·015 in. On removal of the load the cracks closed completely in the case of NR 1, and to 0·002 in. in the case of R 1 and R 2.

(*b*) *At failure*

27. In both series of tests the calculated failure loads of the beams, using limit analysis and assuming complete redistribution of moments at failure, were found to be a safe estimate of the failure loads actually measured on the beams failing in flexure. The moments of resistance of the critical sections were calculated assuming that (i) the reinforcement reached its yield point at failure, (ii) the average concrete compressive stress at failure was 0·6 × cube strength, and (iii) that the centre of concrete compression was at 0·4 of the depth of the concrete compression zone at failure. The measured and calculated loads on the beams at failure are compared in Table 4.

TABLE 4

Beam No.	Total Load at Failure *P* (Tons)		$\frac{P \text{ measured}}{P \text{ calculated}}$
	Measured	Calculated	
1*	5·16	6·40	0·81
2	6·75	6·40	1·06
3	7·28	6·36	1·14
4	6·34	5·97	1·06
NR 1	22·85	20·66	1·10
R 1	21·07	20·96	1·01
R 2	23·5	20·94	1·12

* Failure by local bond slip

28. In the case of the beams failing in flexure, the limit analysis gives a safe and close estimate of the load at failure. Inspection of Table 4 also confirms that redistribution of design moments does not greatly influence the ultimate load-bearing capacity of continuous reinforced concrete beams and hence their factor of safety.

CONCLUDING REMARKS

29. Redistribution of design bending moments for reinforced concrete continuous beams by amounts up to 25% does not appear to affect adversely

the performance of the beam either in the working-load range or at failure. Cracking and deflexion of beams with redistributed design bending moments is not more severe than that of beams designed for the same load, but using the distribution of bending moments predicted by the elastic theory. The factor of safety against failure of a reinforced concrete continuous beam is unaffected by redistribution of the design bending moments. These remarks apply equally to beams reinforced with ordinary mild-steel bars or with work-hardened twisted steel bars.

30. Increase from 15% to 25% of the adjustment to support bending moments allowed in clause 312 of B.S.C.P. 114 (1957) would lead to still more economic structures, without any loss in structural soundness.

ACKNOWLEDGEMENTS

31. The experimental work described in this paper was carried out in the Concrete Technology Laboratory at Imperial College, by permission of Professor A. L. L. Baker, and with the assistance of various postgraduate students and members of staff, to whom thanks are given.

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The Paper, which was received on 14 April, 1958, is accompanied by one photograph and ten sheets of drawings, from which the Figures in the text have been prepared.

Written discussion on this Paper should be forwarded to reach the Institution by 15 July, 1959, and will be published in or after November 1959. Contributions should not exceed 1,200 words.—SEC.
