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STRESSES IN DEEP BEAMS

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SYNOPSIS

Beams whose depths are comparable to their spans are used in a variety of structures. The distribution of bending and shear stresses in such deep beams departs radically from that given by the ordinary, simple formulas for shallow members. Information on the stresses in continuous, deep beams is available elsewhere, and corresponding information for single-span beams is presented in this paper. Five cases of loading are studied, and, for four of these cases, three different span-to-depth ratios are examined. Distributions and magnitudes of bending and shear stresses are given in graphical and tabular form suitable for direct use in design. Although this information is directly applicable to structures made of homogeneous material, such as steel, their use in connection with reinforced concrete requires some special considerations that are briefly outlined.

INTRODUCTION

A deep beam may be defined as one whose depth is comparable to its span. Beams of this type, both in steel and in reinforced concrete, often arise in the construction of bins, hoppers, or similar structures, as well as in more ordinary construction in foundation walls or in cases in which walls are supported on individual columns or footings. The horizontal or vertical diaphragms used to transmit wind forces in buildings (floors or walls) are frequently of such dimensions as to represent deep beams. In reinforced concrete hipped-plate construction[4] the plates of the structure proper or the supporting diaphragms often fall into this category.

In all these cases, design based on the ordinary, straight-line distribution of bending stresses in shallow beams may be seriously in error, since the simple theory of flexure (according to Navier's hypothesis) takes no account of the effect of the normal pressures on the top and bottom edges of the beam caused by the loads and reactions. The effect of these normal pressures on the stress distribution in deep beams is such that the distribution of bending stresses on vertical sections is not linear and the distribution of shear stresses is not parabolic. Consequently, a transverse section which is plane before bending does not remain approximately plane after bending and the neutral axis does not usually lie at the mid-depth, its position being variable in a span-wise direction.

Since information on the stresses in continuous deep beams have already been made available[5] as a result of the work of F. Dischinger,[6] the purpose of this paper is to present the stresses in single-span beams under various types of loading and to indicate how they differ in distribution and magnitude from those predicted by the ordinary beam formulas. It is assumed that the beam material is homogeneous and isotropic. The beams are analyzed as problems in plane stress, using the finite difference method to solve the differential equation for the stress function. The complete details of these solutions have been given by Mr. Chow,[7] while the finite difference method itself has been fully described and discussed in a previous paper by two of the present writers and G. W. Morgan.[8] For this reason, the method proper will not be described in any detail although the results obtained from it are given in full. Although most of these results are new, some obtained previously by H. Bay[9] have been included for completeness and are so indicated on the corresponding distribution charts.

FORMULAS OF THE FINITE DIFFERENCE METHOD

The area enclosed by the four edges of a deep beam is designated as the x-y plane, and this plane is divided into a network of equal divisions by lines parallel to the edges; h and k are the lengths of divisions in the x and y directions, respectively. Furthermore, $Z_{x,y}$ denotes the ordinate at each net-point (x,y) to a curved surface representing G. B. Airy's stress function $Z = f(x,y)$. Then, the biharmonic differential equation that Airy's stress function must satisfy is equivalent to the following linear equation in terms of Z:

$$Z_{x,y}\left\{6\left(\lambda+\frac{1}{\lambda}\right)+8\right\}-4\left\{(1+\lambda)\left(Z_{x-h,y}+Z_{x+h,y}\right)+\left(1+\frac{1}{\lambda}\right)\left(Z_{x,y-k}+Z_{x,y+k}\right)\right\}+2\left(Z_{x-h,y-k}+Z_{x-h,y+k}+Z_{x+h,y-k}+Z_{x+h,y+k}\right)+\lambda\left(Z_{x-2h,y}+Z_{x+2h,y}\right)+\frac{1}{\lambda}\left(Z_{x,y-2k}+Z_{x,y+2k}\right)=0 \quad (1)$$

in which $\lambda = (k/h)^2$.

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[4] "Hipped Plate Construction," by G. Winter and M. Pei, *Journal*, Am. Concrete Inst., Vol. 18, No. 5, 1947, pp. 505-531.

[5] "Design of Deep Girders," *Pamphlet No. ST 66*, Concrete Information, Structural Bureau, Portland Cement Assn., Chicago, Ill.

[6] "Beitrag zur Theorie der Halbscheibe und des wandartigen Trägers," by F. Dischinger, *Publications*, International Assn. for Bridge and Structural Eng., Zurich, Switzerland, Vol. 1, 1932, pp. 69-93.

[7] "Stresses in Deep Beams," by Li Chow, thesis presented to Cornell University, at Ithaca, N. Y., in 1951, in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

[8] "Analysis of Deep Beams," by H. D. Conway, L. Chow, and G. W. Morgan, *Journal of Applied Mechanics*, ASME, Vol. 18, No. 2, June, 1951, pp. 163-172.

[9] "Uber den Spannungszustand in hohen Trägern und die Bewehrung von Eisenbetontragwänden," by H. Bay, Stuttgart, Germany, 1931, p. 64.

The unknown Z-values are determined by solving the set of simultaneous linear equations obtained from the application of Eq. 1 to each net-point. The normal stress σ at any point (x,y) can then be calculated by means of the following formula,

$$\sigma_x = \frac{\partial^2 Z}{\partial y^2} = \frac{Z_{x,y-k} - 2\,Z_{x,y} + Z_{x,y+k}}{k^2} \qquad \text{(2a)}$$

$$\sigma_y = \frac{\partial^2 Z}{\partial x^2} = \frac{Z_{x-h,y} - 2\,Z_{x,y} + Z_{x+h,y}}{h^2} \qquad \text{(2b)}$$

The shear stress τ at the point is

$$\tau_{xy} = -\frac{\partial^2 Z}{\partial x\,\partial y} = \frac{(Z_{x-h/2,y+k/2} + Z_{x+h/2,y-k/2}) - (Z_{x-h/2,y-k/2} + Z_{x+h/2,y+k/2})}{h\,k} \qquad \text{(3)}$$

It should be noticed that Eqs. 2(a) and 2(b) give the normal stresses on the sections along the lines of division of the network, whereas Eq. 3 gives the shear stresses on sections midway between the lines of division, due to the presence of $h/2$ and $k/2$ in the subscripts of Z.

Types of Loading

Five types of loading have been analyzed by the finite difference method and are indicated in Fig. 1. These loadings were chosen to represent, in a general manner, the most common types occurring in practice. Each of the first four types (Fig. 1(a) to Fig. 1(d)) were investigated for three values of the height-to-span ratio, namely, $H/L = \frac{1}{2}$, 1, and 2. The results presented herein utilizing the loading of Fig. 1(a) (with $H/L = \frac{1}{2}$ and 1) and Fig. 1(c) (with $H/L = 1$ and 2) are taken from the analyses made by Mr. Bay, who also used the finite difference method.

For the loading of Fig 1(e), the central portion of the beam is subject to pure bending, and the simple flexure theory predicts zero shear stresses throughout this portion. However, this is not true for deep beams, except for the midspan section in which the shear stresses are zero by symmetry. The case of $H/L = 1$ was, therefore, analyzed to show this deviation from the usual assumption.

For all the cases presented in this paper, $h = L/6$ and $k = H/6$ were used in constructing the network of divisions. However, Mr. Bay used $h = L/4$ and $k = H/4$ for the case of $H/L = \frac{1}{2}$ under the loading of Fig. 1(a) and $h = L/6$ and $k = H/5$ for the case of $H/L = 2$ under the loading of Fig. 1(c). A somewhat better accuracy was, thereby, achieved in the writers' solutions than in those of Mr. Bay.

Results and Discussion

Figs. 2(a), 3(a), 4(a), and 5(a) show the bending stress distributions at the midspan section of a beam under the loadings of Fig. 1(a) to Fig. 1(d). The stress shown is unit stress, being in terms of beam width, b. The midspan section was chosen for consideration since it is the section of maximum bending

moment. It is seen that, for all types of loading, when $H/L = \frac{1}{2}$ the stress curve agrees reasonably well with the linear distribution of the simple flexure theory. The deviation of the curves from the linear distribution becomes increasingly pronounced as the height-to-span ratio increases. The maximum

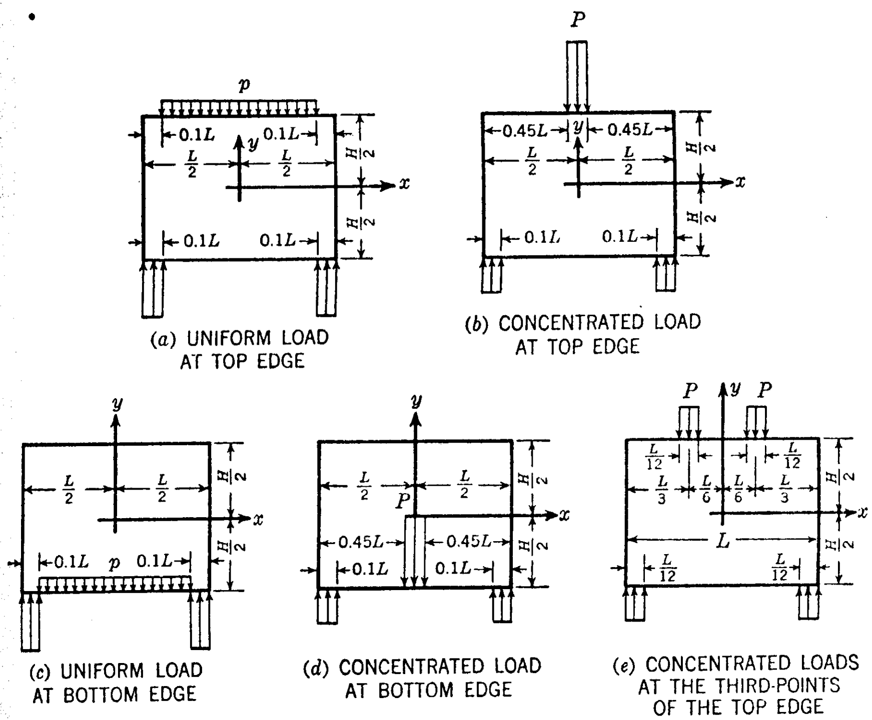

Fig. 1.—Various Types of Loading

bending stress for values of $H/L \geqq 1$ is considerably greater than that predicted by the linear theory, but the stress decreases very rapidly with increasing distance from the edge of maximum stress. There are three points of zero bending stress when $H/L = 2$ with the load acting along the top edge, although the upper half is almost free from bending stress when the load acts along the bottom edge.

Figs. 2(b), 3(b), 4(b), and 5(b) give the shear stress distributions at the section $x = L/4$ under the loading shown in Fig. 1(a) to Fig. 1(d). The total shear force at this transverse section is a maximum for the cases shown in Fig. 1(b) and Fig. 1(d), but it is less than the maximum value that occurs at the section $x = 0.4\,L$, for the cases shown in Fig. 1(a) and Fig. 1(c). Deviations from the simple theory are always present for sections near the loads and reactions, no matter what value the height-to-span ratio may be. These variations are the result of the local influence of the loading and occur also in shallow beams. For this reason, it is preferable to consider a section that is at some distance from the supports for investigating the validity of the simple theory. Moreover, the

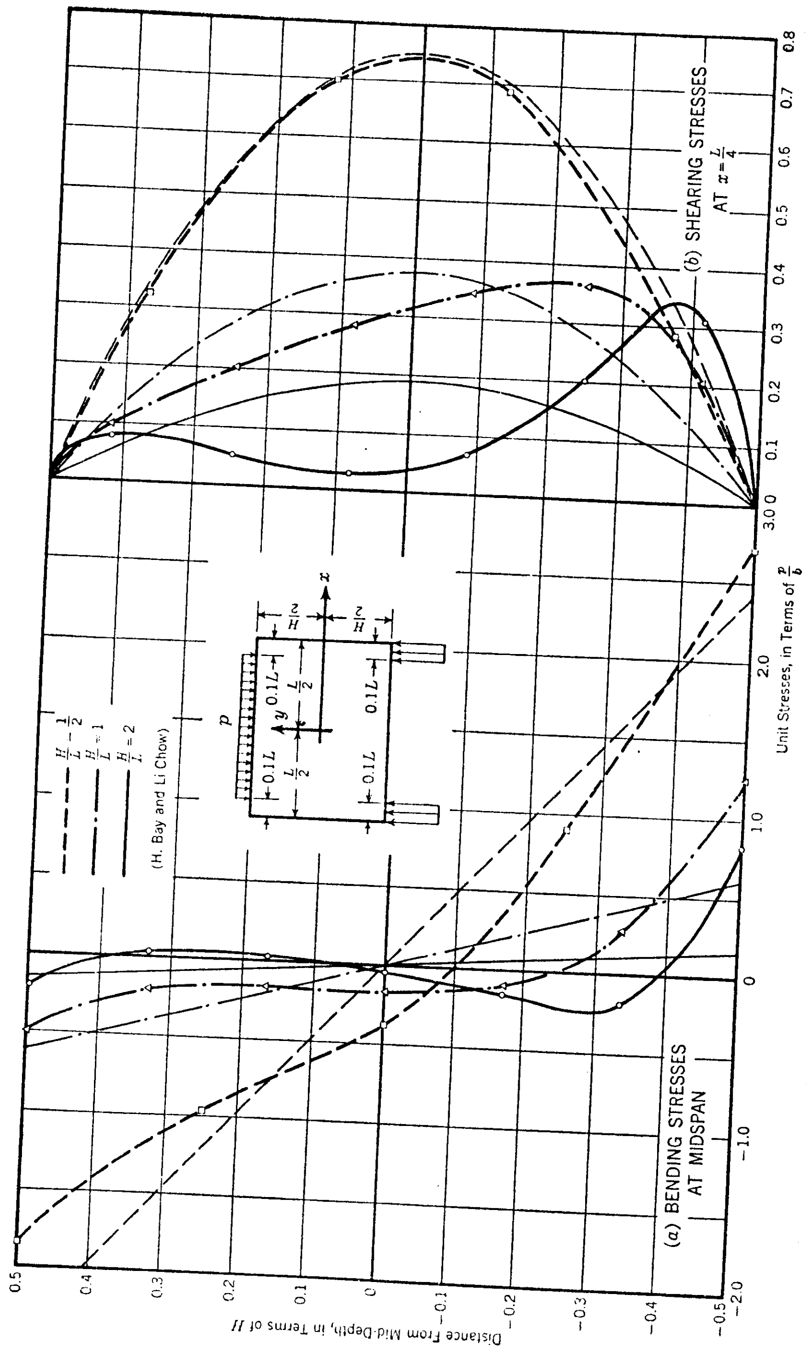

FIG. 2.—STRESSES IN BEAMS HAVING UNIFORM LOADING AT THE TOP EDGE

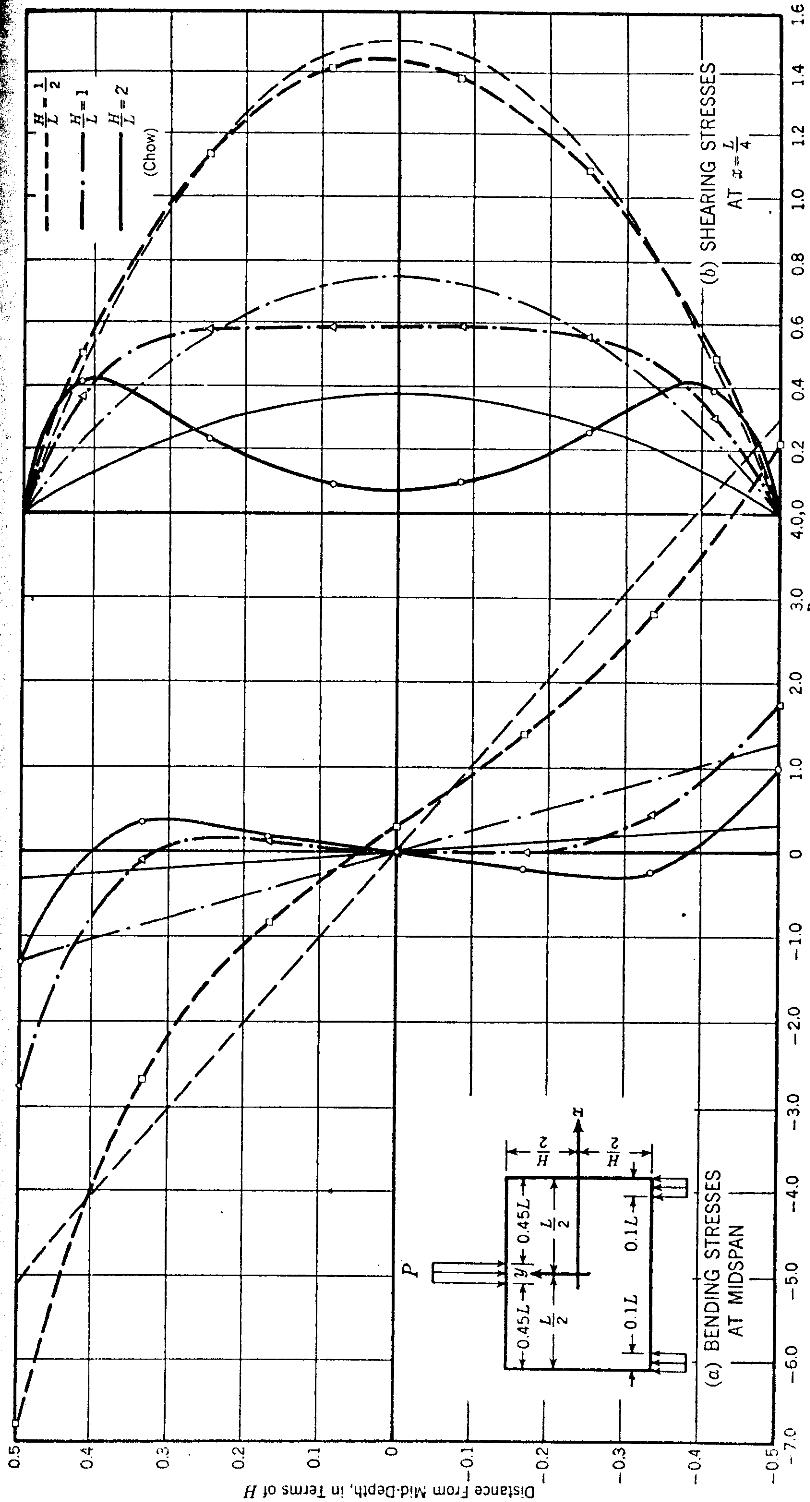

FIG. 3.—STRESSES IN BEAMS HAVING CONCENTRATED LOADING AT THE TOP EDGE

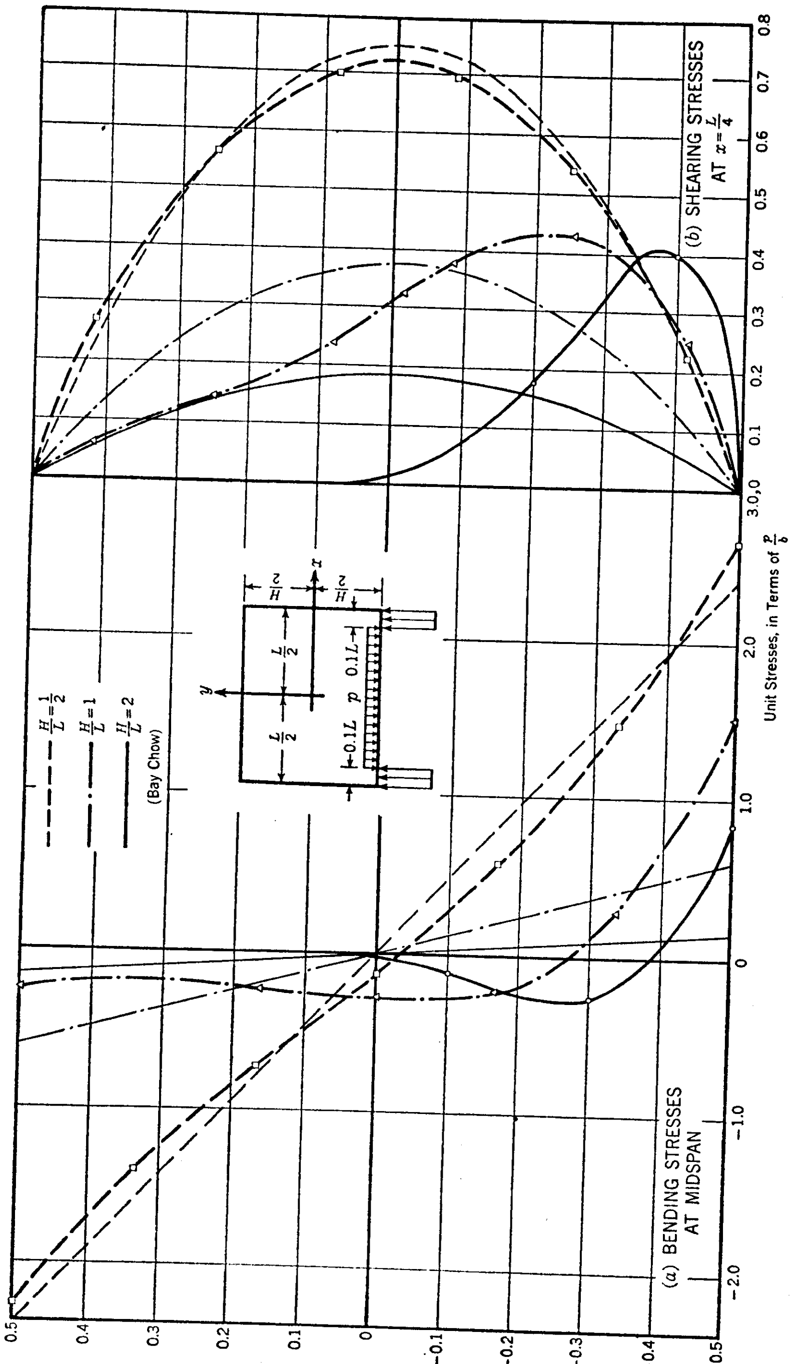

FIG. 4.—STRESSES IN BEAMS HAVING UNIFORM LOADING AT THE BOTTOM EDGE

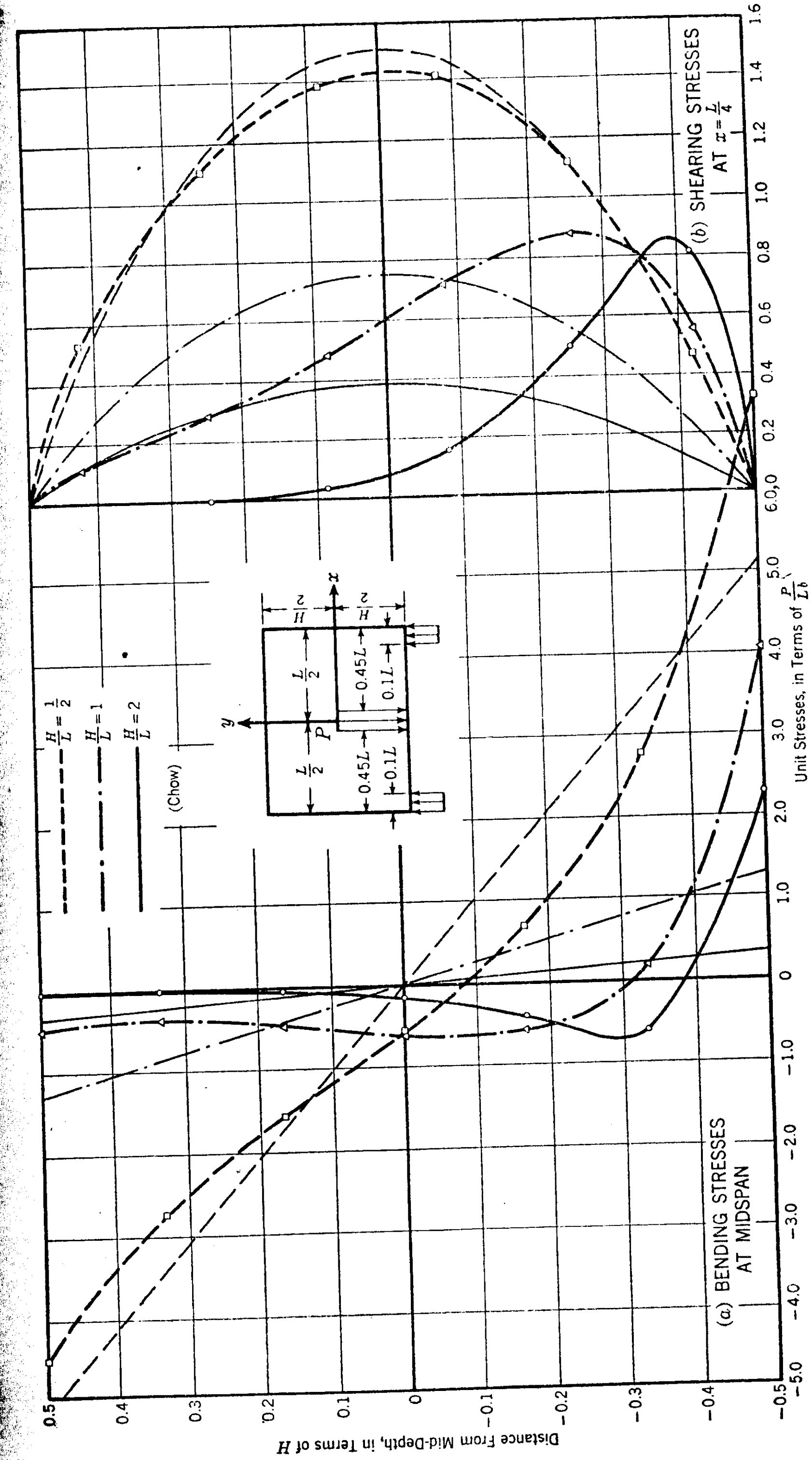

FIG. 5.—STRESSES IN BEAMS HAVING CONCENTRATED LOADING AT THE BOTTOM EDGE

stress values obtained from the finite difference method are usually inaccurate for regions in which abrupt changes of stress and boundary conditions take place. At sections in which the total shear force varies suddenly, the shear stresses computed by this method are of little value.

It is seen from Figs. 2(b), 3(b), 4(b), and 5(b) that the shear stress curves differ radically from the conventional parabolic form when the height-to-span ratio is great. Moreover, the shear stress vanishes in the upper half of the depth, for the cases of $H/L = 2$ with loads acting at the bottom edge. It may also be observed that the areas under these shear stress curves correspond to forces that are somewhat smaller than the actual shear force at the section. This is caused by the inaccuracy of the finite difference method. For this reason, Figs. 2(b), 3(b), 4(b), and 5(b) may be taken as indicating only the probable variations in shear stress and, in so far as actual magnitudes are concerned, these figures must be used with some discretion. However, the distributions of bending stresses given by the finite difference method are known to be quite accurate, and these stresses are probably the most important from a practical point of view.

Fig. 6(a) shows the bending stress distribution at sections $x = 0$ and $x = L/6$ under the loading of Fig. 1(e), with $H/L = 1$. These sections have nearly equal bending moments (0.292 $P\,L$ for $x = 0$ and 0.281 $P\,L$ for $x = L/6$), but their stress curves are quite different. The maximum compressive bending stress at the upper edge for the section ($x = L/6$) is about twice that for the section ($x = 0$), although the latter section has a bending moment that is slightly greater. This indicates the presence of large localized stresses in the vicinity of concentrated loads. It may also be pointed out from an examination of Fig. 6(a) that the shapes of the bending stress curves for two adjacent transverse sections are not similar to each other and their neutral axes do not necessarily coincide.

Fig. 6(b) shows the shear stress distributions at the sections $x = L/12$, $L/4$, and $5\,L/12$ for the loading of Fig. 1(e), with $H/L = 1$. The peaks of the stress curves are near the top edge for sections near the central portion of the span, but these peaks are near the bottom edge for sections near the supports. Although the total shear force is zero at the section ($x = L/12$), the shear stresses do not vanish in this section. It has been shown[7] that in the neighborhood of a concentrated load, localized shear stresses are induced near the discontinuity of the large normal pressure. Thus, localized shear stresses are produced at the section $x = L/12$, in the vicinity of the concentrated load, and shear stresses of opposite sign must also be induced in other parts of the same section, since the total shear force is zero.

The inaccuracy of the shear stress values in Fig. 6(b) is more pronounced than that in Figs. 2(b), 3(b), 4(b), and 5(b), since each of the three sections considered is located close to a point of sudden change of shear force. As may be observed from Fig. 6(b), the algebraic sum of the areas under the curve for the section $x = L/12$ is not zero and the areas under the curves for the other two sections correspond to forces that are appreciably less than the total shearing force P indicated by the area under the parabola.

All the stress values given in Figs. 2 to 6 are either in terms of p/b, in which p is the uniform load per unit length of the beam and b is the width of the beam, or in terms of $P/(L\,b)$, in which P is the concentrated load and L is the length of the beam—according to the type of loading indicated in each of the figures.

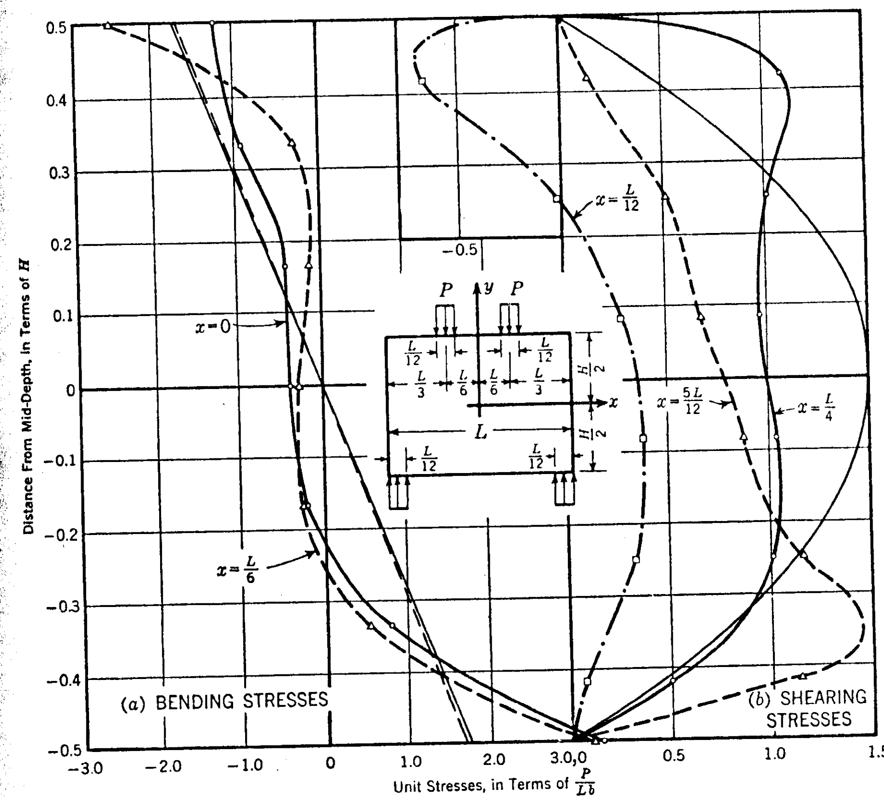

FIG. 6.—STRESSES FOR BEAMS WITH $\frac{H}{L} = 1$ AND HAVING CONCENTRATED LOADS AT THE THIRD POINTS OF THE TOP EDGE

By means of the principle of superposition, it is possible to compute the stresses for any combination of the given loadings, such as for uniform loading with a concentrated load at the center of the span. For example, let it be required to find the extreme fiber stresses at the midspan section of a single-span deep beam for which the following data are given: Length of support for reaction or concentrated load = 3 ft; span (L) = 30 ft; height of beam (H) = 30 ft; width of beam (b) = 1.25 ft; uniform load along top edge = 30,000 lb per lin ft = 2,500 lb per in.; and concentrated load at center of top edge = 100,000 lb. Since $H/L = 1$, the stress coefficients selected from Figs. 2(a) and 3(a) are − 0.50 and + 1.24 for the uniform loading and − 2.74 and + 1.74 for the concentrated load. The plus sign indicates tension and the minus sign, compression. The maximum and minimum bending stresses in pounds per square inch for the combined loading are then

$$(\sigma_x)_{y=+\frac{H}{2}} = (-0.50) \times \frac{2{,}500}{15} + (-2.74) \times \frac{100{,}000}{15(30 \times 12)} = -83 - 51 = -134 \text{ lb per sq in.}$$

$$(\sigma_x)_{y=-\frac{H}{2}} = 1.24 \times \frac{2{,}500}{15} + 1.74 \times \frac{100{,}000}{15(30 \times 12)} = 207 + 32 = +239 \text{ lb per sq in.}$$

Mention may be made here of another method for approximate determination of the stresses in a deep beam, suggested by A. P. Sinitsyn.[10,11] This method is based on the similarity between the differential equation for the Airy stress function and that for the deflection of a plate under transverse loading. Use is made of this analogy, but, instead of solving for the analogous plate deflection from the differential equation, simplifying assumptions enable these deflections to be determined in an approximate manner that, essentially, neglects the torsional plate moments. The results undoubtedly are in error at certain places in which the assumptions made have a significant effect in changing the true picture of the plate action. The accuracy of the results depends on the position of the points selected, the ratio of H/L, and the type of loading. This may be seen from the following comparison between the results of Sinitsyn's method and those given by the finite difference method. The discrepancy is substantial in certain cases. Stresses in Table 1 are in units of p/b and the midspan section of a beam is considered.

TABLE I.—COMPARISON OF RESULTS

Values of H/L	$(\sigma_x)_{y=+\frac{H}{2}}$		$(\sigma_x)_{y=-\frac{H}{2}}$	
	Sinitsyn's solution	Finite difference method	Sinitsyn's solution	Finite difference method
(a) UNIFORM LOAD p AT BOTTOM EDGE				
½	−2.38	−1.82	+2.69	+2.71
1	−0.53	−0.50	+1.77	+1.24
(b) UNIFORM LOAD p AT BOTTOM EDGE				
½	−2.33	−2.26	+2.62	+2.65
1	−0.32	−0.24	+1.52	+1.53

APPLICATION TO THE DESIGN OF REINFORCED CONCRETE STRUCTURES

For beams of homogeneous material, a determination of the bending and shear stresses by means of Figs. 2 to 6 gives sufficient information for design purposes. Reinforced concrete, on the other hand, is not a homogeneous material, so that one of the basic assumptions of the preceding data is not satisfied. The additional fact that the tension zone of reinforced concrete beams must be considered cracked, modifies the stress distribution even in shallow beams

[10] "Approximate Analysis of Beam-Walls" (in Russian), by A. P. Sinitsyn, *Proyect y Standart*, No. 5, 1935, p. 21.
[11] *Ibid.*, No. 10, p. 24.

(Fig. 7). The analysis is simplified for such beams, however, by the reasonable validity of the assumption that plane cross sections remain plane despite cracking. For deep beams, on the other hand, this assumption cannot be made. The stress distribution in such reinforced concrete members must be expected to differ from that given by Figs. 2 to 6 on two counts: (1) The nonhomogeneity of material; and (2) the cracking of the tension zone. A rigorous, theoretical analysis of the stresses in such beams is hardly feasible. For this reason, no unique and strictly justified design procedure can be proposed. Some discussion of this problem and suggestions regarding tension and shear reinforcement are given in the following sections.

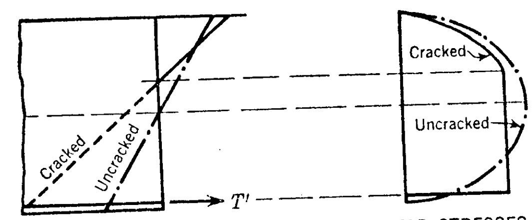

FIG. 7.—EFFECT OF CRACKING ON STRESS DISTRIBUTIONS IN A SHALLOW BEAM

Tensile Reinforcement.—The Portland Cement Association,[5] following European practice, suggests that the required steel area A_s be determined from

$$A_s = \frac{T}{f_s} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which T is the total tension force computed from the homogeneous beam analysis (that is, Figs. 2 to 6) and f_s is the allowable unit stress of the steel. The entire tension reinforcement is located near the tension (bottom) edge. It must be realized that this procedure, although possibly safe in regard to strength, destroys completely the very assumptions on which the analysis is based and from which the value of T is determined. Indeed, by concentrating the tension reinforcement near the edge, the total tension force in the cracked section is forced to shift from the centroid of the tension areas of Figs. 2 to 6 to the centroid of the reinforcement (Fig. 8(a)). This is likely to increase the

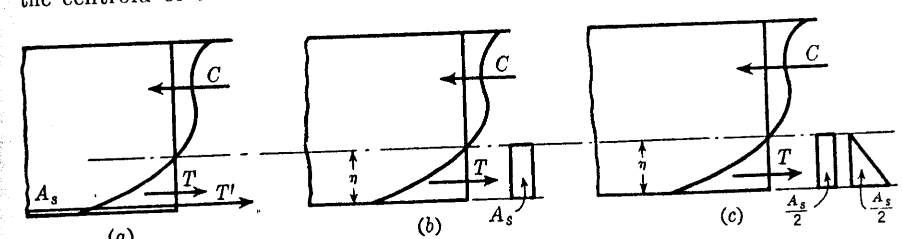

FIG. 8.—PROPOSED ARRANGEMENTS OF TENSILE REINFORCEMENT

effective internal lever arm although such a statement cannot be made with certainty for lack of rigorous, analytical support. If such increase does occur, it implies, of course, a decrease of the actual force T' in the reinforcement as compared to its computed value T and a corresponding decrease of the total compression force, since the couple formed by these forces equals the given static moment. As a consequence, the resulting distribution of both normal and shear stresses is likely to be entirely different from that obtained from the analysis of the homogeneous beam. Even if a change of lever arm would not occur, the stress distribution would still be strongly affected by the forced shift of the tensile force to the edge since, in this case, the centroid of the compression area would have to shift by the same amount in the same direction to maintain equilibrium, with a consequent radical change in the distribution of the compression and shear stresses.

A further, practical argument against locating the reinforcement in this manner concerns the type and extent of cracking likely to occur. Cracks that begin at the tension edge would then be intercepted by reinforcement only near the edge, without any further reinforcement counteracting their growth in length and width up to the neutral axis. Since this axis, even in the homogeneous beam, is located at a distance from the tension edge varying from one half to one ninth of the depth—and likely to be larger yet in the nonhomogeneous, cracked beam—cracks of very considerable extent and width result in large, wall-like beams. To reduce these cracks additional distributed reinforcement at least of the amount usually prescribed against shrinkage would be called for over the entire tension zone.

Another scheme of reinforcing that could be rationally justified is one that would convert the beam as nearly as possible into a homogeneous beam. This can be achieved by distributing horizontal reinforcement uniformly throughout the tension zone to provide equal resistance and rigidity for every unit of depth throughout that zone, which is the essential prerequisite of "nearly homogeneous action." It is obvious that such distributed reinforcement would minimize cracking and would result in a location of the resultant of the steel stresses very close to, if not coincident with, that of the theoretical location of T (Fig. 8(*b*)).

To determine the amount of reinforcement required in this case, use can be made of the fact that from Figs. 2 to 6 the tensile stresses are seen to be nearly linearly distributed. Hence, with uniform steel distibution, the average steel stress for all practical purposes equals half that in the bar closest to the edge. The latter, as usual, must be limited to the allowable steel stress f_s. Hence

$$A_s = \frac{T}{f_{av}} = \frac{T}{0.5 f_s} = \frac{2\,T}{f_s} \qquad (5)$$

Comparison with Eq. 4 indicates that, for this manner of reinforcement, exactly twice as much steel would be required as in the first case, in which the entire steel was concentrated near the tension edge. It is likely, therefore, that such uniform reinforcement is overconservative and uneconomical, since the part of the steel near the neutral axis is stressed very little.

For an apparently sensible compromise between these conflicting requirements a different approach suggsets itself.

The total required steel area may be determined from

$$A_s = \frac{1.5\,T}{f_s} \qquad (6)$$

Half of this area is uniformly distributed throughout the tension zone, while for the other half the spacing between rods is increased linearly with increasing distance from the tension edge, as schematically indicated in Fig. 8(*c*). Such distribution evidently provides sufficient steel at all levels to counteract excessive cracking and to secure at least some measure of homogeneous action. It can also be shown that the outermost rods of this arrangement will not be overstressed. Indeed, the arrangement of Fig. 8(*b*), which provides for the required stress f_s in the outer rods, results in an amount of steel per vertical inch at the tension edge, equal to

$$a_s = \frac{2\,T}{f_s\,\eta} \qquad (7)$$

in which η is the distance to the neutral axis from the tension edge (see Fig. 8). For the arrangement of Fig. 8(*c*) and Eq. 6, the amount of reinforcement per vertical inch at the tension edge, where tensile stresses are highest, is easily computed from the separate contributions of the two total areas $\frac{A_s}{2}$ each, and equals

$$a_s = \frac{0.75\,T}{f_s\,\eta} + \frac{2 \times 0.75\,T}{f_s\,\eta} = \frac{2.25\,T}{f_s\,\eta} \qquad (8)$$

Comparison of Eqs. 7 and 8 shows that this arrangement provides a steel density 12.5% larger than that required in scheme Fig. 8(*b*) to insure the edge steel against overstressing. (If this suggested scheme is followed, obviously no actual distinction need be made between the rods of the two differently distributed areas $\frac{A_s}{2}$. Since, at the neutral axis, $a_s = \frac{0.75\,T}{f_s\,\eta}$, comparison with Eq. 8 shows that such a steel distribution is achieved if the total area A_s is so arranged that the area per vertical inch or foot at the tension edge is three times that at the neutral axis, with reasonably linear transition between these two extremes.) It is not likely that the steel area required by this alternative is significantly larger than that for Eq. 4 and for Fig. 8(*a*), if, for the latter arrangement, the required additional shrinkage steel is added to the computed reinforcement A_s.

Since the use of any of these formulas for A_s requires that the magnitude of the total tensile force T and the location of the neutral axis be known, these values, measured from the stress curves, are given in Table 2.

TABLE 2—TOTAL TENSILE FORCE T AND DISTANCE η OF THE NEUTRAL AXIS FROM THE TENSION EDGE FOR SECTION $x=0$

Types of loading	$\frac{H}{L}$	T	$\frac{\eta}{H}$
Uniform load p at top edge (Fig. 1(*a*))	½	$0.510\ p\ H$	0.400
	1	$0.131\ p\ H$	0.255
	2	$0.010\ p\ H$ $0.045\ p\ H$	{0.890 {0.583 0.111
Concentrated load P at top edge (Fig. 1(*b*))	½	$1.125\ P\ H/L$	0.547
	1	$0.022\ P\ H/L$ $0.196\ P\ H/L$	{0.820 {0.495 0.336
	2	$0.085\ P\ H/L$ $0.047\ P\ H/L$	{0.900 {0.505 0.117
Uniform load p at bottom edge (Fig. 1(*c*))	½	$0.549\ p\ H$	0.467
	1	$0.161\ p\ H$	0.228
	2	$0.060\ p\ H$	0.125
Concentrated load P at bottom edge (Fig. 1(*d*))	½	$1.097\ P\ H/L$	0.418
	1	$0.353\ P\ H/L$	0.186
	2	$0.146\ P\ H/L$	0.115
Two loads P at third points of top edge (Fig. 1(*e*))	1	$0.359\ P\ H/L$	0.278

It is seen from the stress distribution graphs that in some cases a second tensile region exists in addition to the bottom one. The stresses in these regions are often too small to cause cracks, making reinforcement unnecessary. If these stresses exceed the tensile strength of concrete, the steel area required can be easily computed from the corresponding value of T.

Shear Reinforcement.—The question of shear reinforcement is rather involved, even for shallow beams. The conventional procedure of designing for shear reinforcement is not theoretically exact, since it is merely an approximate means of accounting for the inclined principal tensile stresses in the concrete. In fact, beams do not fail by direct shear but by inclined tension induced by shear and normal stresses. The allowable shear stress v_c for concrete is determined empirically in such a manner that the value is low enough to insure against failure by inclined tension. Therefore, the allowable values of v_c are known to be considerably below the safe working stress of concrete in direct shear.[12]

For shallow, homogeneous beams in which the effect of σ_y is customarily neglected, the principal tensile stress σ_1 is expressed by

$$\sigma_1 = \frac{\sigma_x}{2} + \frac{1}{2}\sqrt{\sigma^2_x + 4\,\tau^2_{xy}} \quad \text{(9)}$$

and the angle of inclination θ between σ_1 and σ_x is given by

$$\tan 2\,\theta = \frac{2\,\tau_{xy}}{\sigma_x} \quad \text{(10)}$$

At the neutral axis $\sigma_x = 0$ so that $\sigma_1 = \tau_{xy}$ and $\theta = 45°$. The same condition is assumed to be true in a reinforced concrete beam for any point below the neutral axis since, in view of cracking, σ_x is assumed to be zero throughout the tension part of the beam. Thus, the maximum inclined tension is, with sufficient accuracy, equal to the maximum shear stress, and web reinforcement is designed to resist the difference between this maximum shear stress and the amount of shear assigned to the concrete. It is well known that this method is a somewhat crude approximation and even slightly contradictory in itself, but it leads to a workable and safe design method for shallow beams.

However, even for shallow beams, Mr. Bay[13,14,15] has shown that the customary shear investigation leads to an overestimate of the magnitude of the maximum principal tensile stress and that the critical section for inclined tension is not at the support but at a distance of approximately 0.65 H from the support. This is true since, near a support, the values of the vertical compressive stresses σ_y are large and cannot be neglected. Therefore, Eqs. 9 and

[12] "Design of Concrete Structures," by L. C. Urquhart and C. E. O'Rourke, McGraw-Hill Book Co., Inc., New York, N. Y., 4th Ed., 1940, p. 104.

[13] "Die schiefen Hauptzugspannungen beim Eisenbetonbalken," by H. Bay, *Ingenieur-Archiv*, Vol. 4, 1933, p. 244.

[14] "Scherbeanspruchung und Scherfestigkeit beim Beton, by H. Bay, *ibid.*, Vol. 14, 1943, pp. 267-276.

[15] "Der Einfluss der lotrechten Pressungen auf die Hauptzugspannungen beim Eisenbetonträger," by H. Bay, *Beton und Eisen*, Vol. 22, 1933, pp. 239-241.

10 are replaced by

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \frac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\,\tau^2_{xy}} \quad \text{(11)}$$

$$\tan 2\,\theta = \frac{2\,\tau_{xy}}{\sigma_x - \sigma_y} \quad \text{(12)}$$

As seen from these equations, the presence of the compressive stress σ_y in the vicinity of the supports will reduce the magnitude of σ_1 and also make θ considerably less than 45°.

The principal tensile stresses are, therefore, nearly horizontal and may be carried by the logitudinal steel. Mr. Bay confirmed his analysis by tests that showed that the vicinity of the supports was free from cracks at loads far greater than those that had caused inclined cracks to occur in the same beams at points farther removed from the supports. The value of σ_y decreases with increasing distance from the support, so its effect on σ_1 becomes negligible at a distance of about 0.65 H from the support. Hence, σ_1 reaches a maximum at this point. The tests by Mr. Bay also showed that the first inclined crack actually appeared at this place at which σ_1 was a maximum.

The foregoing discussion is presented for the purpose of illustrating some of the difficulties involved in obtaining a logically consistent procedure for designing shear reinforcement. It is obvious that the conventional, semi-empirical method for shallow beams cannot be applied to deep beams in which the magnitude of σ_1 is greatly affected by the presence of σ_y throughout the beam, in a manner similar to that just discussed for the vicinity of the supports of shallow beams. For these reasons the shear stress cannot be regarded as a reliable measure of inclined tension in deep beams, and accurate information could be obtained only by actually computing principal tension stresses. Since such a procedure would be extremely involved, the approximate method for shear in deep beams, proposed by the Portland Cement Association,[5] appears sensible. It is suggested that, for $H/L > 0.4$, the allowable shear stress for deep beams be increased to $\dfrac{v_c\left(1 + 5\dfrac{H}{L}\right)}{3}$ in which v_c is the value allowed for shallow beams. This accounts for the fact that, because of the presence of vertical compression σ_y, the inclined tension σ_1 is smaller than the maximum shear τ_{max}. Instead of computing this lower inclined tension and comparing it with v_c, the Portland Cement Association method proposes raising the allowable shear stress and comparing it with τ_{max}. The effect is identical, of course, at least qualitatively.

It is further recommended in the previously mentioned publication that the maximum shear be computed from $\tau_{max} = \dfrac{8\,V}{7b\,H}$, that is, from the usual, approximate formula for shallow beams. This procedure may be justifiable for lack of better information if the beam is reinforced according to Fig. 8(*a*) (with reinforcement at the tension edge) since, in that case, the actual stress distribution is highly uncertain. However, if reinforcement is arranged to approach more closely the condition of homogeneity (such as in Fig. 8(*b*) or 8(*c*)), the shear distribution can be assumed to be fairly close to that in a homogeneous beam, and τ_{max} can then be taken from the appropriate curves of Figs. 2 to 6.

In many cases τ_{max} will be found to be smaller than the shear value assigned to concrete in deep beams, so that shear reinforcement is not required. If this is not the case such reinforcement must be provided primarily by inclined bars or by a network of horizontal and vertical bars. In contrast to shallow beams, cracks in deep beams are nearly vertical (since principal tensions are nearly horizontal) so that vertical shear reinforcement, such as stirrups, is largely ineffective.

This brief discussion of the questions involved in correctly reinforcing deep concrete beams is not meant to be either exhaustive or authoritative. It is merely intended to point up some of the differences in approach required in designing deep as compared to shallow concrete beams and to suggest methods that seem reasonable to the authors. Experimental investigations in this field would do more to settle the question of the most desirable arrangement of reinforcement than any amount of theorizing.

Conclusions

For shallow beams, the normal pressures on the longitudinal edges have little effect on the stress distributions at sections a small distance from supports or concentrated loads. The stresses at these sections depend only on the bending moment and shear force, and the values computed by the ordinary beam formulas are accurate for practical purposes. The stresses in deep beams, however, depend not only on the bending moment and shear force at a section but also on the variation of normal pressures along the loaded edges. The greater the height-to-span ratio, the more significant becomes the latter factor. Thus, for deep beams in which the load is applied to the bottom edge, the stress distributions are quite different from those caused by the same loading applied at the top edge. In general, the ordinary beam formulas may be considered as valid for computing stresses in beams whose height-to-span ratio is less than one half and in which the sections under study are not too near to loads and reactions.

Analytical results for the stresses in deep beams of homogeneous material are presented and can be applied directly to the design of structures made of such material. The design of deep reinforced concrete beams, on the other hand, involves the difficulty of dealing with a slightly cracked, nonhomogeneous material. A rigorous theoretical analysis of this situation is hardly possible. The stresses obtained from analyses based on the assumption of homogeneity may serve as a reasonable guide for estimating the actual stresses in the cracked state of the nonhomogeneous beam, provided the beam is so reinforced as to approach homogeneous performance. Suggestions regarding tensile and shear reinforcement have been presented to achieve such performance and to limit the extent of cracking.

Acknowledgment

The need for more detailed data for the design of deep beams was established in discussions of the Subcommittee on Hipped Plate Structures (Committee on Masonry and Reinforced Concrete, ASCE Structural Division), of which two of the writers are members.

DISCUSSION

Arturo M. Guzmán[16] and Cesar J. Luisoni[17].—Special interest has been aroused by the subject because of the importance of the deep beam as a strong structural element in reinforced concrete industrial buildings and in large residential blocks. In France, numerous detailed specifications have been issued by the government on this matter.[18]

The analysis of a single-span deep beam is particularly difficult. For their calculations, the authors use the method of finite differences, and the agreement of the approximation depends, of course, on the density of the network that has been adopted. For the case $H = L$ with supporting width 0.1 L and the load on the top edge, there exist complete photoelastic tests using the very exact, purely optical, Favre method.[19] In the central section, for the maximum bending stress, σ_x, the difference is 20% at the bottom edge; and the finite differences method, with the network width as indicated by the authors, yields smaller values than the experimental method.

A more accurate approximation, involving less calculation,[20,21] is possible by using fifth-degree polynomials for the Airy stress function and, therefore, third-degree polynomials for σ_x, the coefficients for the polynomials being determined by the Galerkin variational method. This solution has then been extended to several cases of load,[22,23] with tabular data being given for each. In the case analyzed by the authors, using finite differences, fifteen equations with as many unknowns must be solved, whereas with the Galerkin variational method used by the writers, only four equations with as many unknowns are required, the agreement with the photoelastic test being very good.

For the case $H = 1.5\ L$, uniformly loaded, the writers have used the finite differences method with a very dense network, $h = k = L/8$ or 44 points, solving the equations by the Southwell relaxation method[22,23] and using the so-called block relaxation to speed up the convergence; even so, however, the calculation work is considerable. Nevertheless, differences of about 10% to 20% as compared with experimental values[24] always arise. The rectangular network used by Mr. Bay[9] and the authors is not to be recommended as it

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[18] "Traité de Beton Armé," by A. Guerrin, *Règles B. A. 45 du Ministère de la Reconstruction et de l'Urbanisme (1945–1948)*, Paris, France, Vol. II, 1951, pp. 208–216.

[19] "Ensayo fotoelástico de una viga de gran altura," by C. A. Sciammarella and M. A. Palacio, *Ciencia y Técnica*, Buenos Aires, Argentina, Vol. 113, No. 569, 1949.

[20] "Solución variacional del problema de la viga rectangular simplemente apoyada de gran altura," by A. M. Guzmán and C. J. Luisoni, *ibid.*, Buenos Aires, Argentina, Vol. 111, No. 555, 1948.

[21] *Applied Mechanics Reviews*, ASME, Vol. 1, November, 1948, Revs. 1608.

[22] "Sobre la viga simplemente apoyada de gran altura. Teoria y experimentación," by A. M. Guzmán and C. J. Luisoni, *Ciencia y Técnica*, Buenos Aires, Argentina, Vol. 114, No. 576, 1950.

[23] *Applied Mechanics Reviews*, ASME, Vol. 3, October, 1950, Revs. 1867.

[24] "Ensayo fotoelástico de una viga de gran altura," by C. A. Sciammarella and M. A. Palacio, *Ciencia y Técnica*, Buenos Aires, Argentina, Vol. 117, No. 588, 1951.

increases the error. This defect is discernible in Fig. 4 for $H = 2L$, in which the surface of the compression zone is about 70% superior to the tension zone for σ_x, whereas both must be equal for calculation control and static conditions.

For uniform load, the maximum bending stress at the bottom edge does not occur in the middle but on the sides. This fact cannot appear in finite differences, but is revealed in the variational method, and has been confirmed by photoelastic tests.

The behavior of tensile reinforcement, for single-span or two-span reinforced concrete deep beams, has not been studied extensively. Special mention should be made of a contribution by H. Nylander and H. Holst[25] that is not sufficiently known, and a recent discussion in which experiments by Garcia Olano-Fliess[26] are cited. The Swedish authors[25] give interesting conclusions regarding the cracking mechanism of deep beams with a load on the top edge, which can be forecast by studying theoretically derived isostatics or by photoelasticity. Messrs. Nylander and Holst[27] state:

> "* * * after the formation of tensile cracks the member acts as a straining trestle-work provided with tension rods, and the load-bearing capacity is therefore dependent on the anchorage of the bending tensile reinforcement. For this reason, the shear reinforcement should be proportioned so as to provide safety against the formation of detrimental cracks."

Tests[26] for a single-span deep beam with a load at the top edge, and for $H = 1.5\ L$, reveal that the failure occurs when the member has lost its resistance to compression in the supports and that the tension in the reinforcement at failure is very low. The cracks always start on the interior supporting edge and spread upward, tending to form arches. The function of the reinforcement is then to act as a tension rod, and under these conditions the required section seems to be very much smaller than the one calculated by prior methods. Special reinforcement should be provided at the supports, taking into account the local concentration of stress. The usual criterion seems then to be uneconomical. The proposed criteria have been applied in recently-constructed industrial buildings in Argentina,[24] and a satisfactory behavior has been observed, with a considerable saving of steel.

William A. Conwell,[28] M. ASCE.—The authors are to be commended for bringing to the publications of the ASCE the results of problems of great interest to the structural engineer. However, a real appreciation of what has been done requires reference to the work of two of the present authors and Mr. Morgan,[8] in which they outline their methods. The writer believes that only with this work as a background can one assess the value of this paper. It would have been most fortuitous had both works been published under one cover.

[25] "Några Undersökningar Rörande Skivor Och Höga Balkar Av Armerad Betong," by H. Nylander and H. Holst, *Transactions*, Royal Technical Univ., Stockholm, Sweden, No. 2, 1946.

[26] "Vigas de gran altura: Algunos criterios para su cálculo y disposición de la armadura," by C. A. Sciammarella, *Hormigón Elástico*, Buenos Aires, Argentina, December, 1951.

[27] "Några Undersökningar Rörande Skivor Och Höga Balkar Av Armerad Betong," by H. Nylander and H. Holst, *Transactions* Royal Technical Univ., Stockholm, Sweden, No. 2, 1946, pp. 51–52.

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There are two distinct goals in this type of work. They are: (1) To provide methods with which the engineer can obtain, quickly, approximate solutions which are an aid to his judgment and better than rule-of-thumb procedures; and (2) to give the engineer data, based on rigorous research, upon which he can base his designs. The paper, with its predecessor,[8] accomplishes the first end. Although it may not fully attain the second, particularly in the field of shear stresses (which can be important), it certainly indicates the direction in which considerably more work is needed.

One of the striking results of the paper is the close agreement between the authors' diagrams for shear and bending stresses for $\frac{H}{L} = \frac{1}{2}$, and those stresses obtained by the usual theory of flexure (see Figs. 2, 3, 4, and 5). It should serve to indicate that subsequent work might be directed toward problems having values of $\frac{H}{L}$ greater than $\frac{1}{2}$.

The authors indicate that the "*** shear stress curves correspond to forces that are somewhat smaller than the actual shear force at the section" (see under the heading, "Results and Discussion"). By reading values from the shear curves for $\frac{H}{L} = 2$, the writer found errors in total shear of 8.4%, 11.8%, 15.2%, and 9.2% for Figs. 2(*b*), 3(*b*), 4(*b*), and 5(*b*), respectively. These percentages of error in total shear indicate that there are probably still greater errors in the shear stress at any point. Although the errors may seem large, the data remain valuable as an aid to judgment; and, although the authors charge the errors to the "*** inaccuracy of the finite difference method" (see under the heading, "Results and Discussion"), there is no reason why the errors cannot be reduced substantially by using the finite difference method with a finer net. It may be necessary to introduce the finer net only in a part of the beam by methods developed by George H. Shortley, Royal Weller, and Bernard Fried.[29] The finer net would seem to be worthwhile in the region of sharp curvature of the shear stress curve, for example, in the region from $-0.3\ H$ to $-0.5\ H$.

A valuable addition to the paper (perhaps it could be included in the closing discussion) would be a contour plot of the principal stresses for at least one of the examples. Engineers are familiar with such plots, based on the usual theory of flexure and made in elementary strength of materials courses. A comparison with a contour plot obtained by the authors' method would be interesting.

It might be questioned whether, in the use of an expression as involved as Eq. 1, the authors have not lost some of the advantages usually attributed to the physical significance of the finite difference method. It is believed that considerable simplification would result in taking $\lambda = 1$, and treating with a special equation any rectangular net elements that might appear at the bound-

[29] "Numerical Solutions of LaPlace's and Poisson's Equations," by George H. Shortley, Royal Weller, and Bernard Fried, *Bulletin No. 107*, Eng. Experiment Station, Ohio State Univ., Columbus, Ohio, September, 1940.

aries.[29] Still further, Eq. 1 is the difference equation equivalent to a fourth-order differential equation. The physical concept of such an equation often staggers the mind of an engineer who finds second-order equations (including partial derivatives) well within his grasp. Furthermore, the equation does not readily lend itself to the use of the numerical methods which can be very powerful in these problems.

The writer suggests the inclusion of a function,

$$u = \frac{\partial^2 Z}{\partial x^2} + \frac{\partial^2 Z}{\partial y^2} \quad \text{(13)}$$

The resulting difference equations would be

$$u_{x,y} = \frac{u_{x-\Delta,y} + u_{x+\Delta,y} + u_{x,y-\Delta} + u_{x,y+\Delta}}{4} \quad \text{(14)}$$

$$Z_{x,y} = \frac{Z_{x-\Delta,y} + Z_{x+\Delta,y} + Z_{x,y-\Delta} + Z_{x,y+\Delta} + u_{x,y}\,\Delta^2}{4} \quad \text{(15)}$$

in which $h = k = \Delta$. The writer cannot say that it would be easier to solve simultaneously twelve equations of the type of Eqs. 14 and 15, rather than six of the more involved type of Eq. 1. However, it is apparent that Eqs. 14 and 15 lend themselves more readily to numerical computations.

As for physical significance, because $\nabla^2 Z = u$ and $\nabla^2 u = 0$, the sum of the angle changes in the x-direction and y-direction on the Z-surface equals u, and the sum of the angle changes in the x-direction and y-direction on the u-surface equals zero.

As an indication of the relative simplicity of the procedure which involves writing difference equations at two levels, the writer has selected from the previous paper by Messrs. Conway, Chow, and Morgan[8] the array of Z-values for the problem treated there, which would go into Eq. 1.

		6.038		
	5.440	4.754	2.778	
3.500	4.000	3.500	2.000	−0.500
	2.560	2.246	1.222	
		0.962		

This array is composed of Z-values in terms of $\frac{P\,a^2}{32}$.

The corresponding sets of values for the two-level procedure at the same point are

Values of Z

	4.754	
4.000	3.500	2.000
	2.246	

Values of $u\,\Delta^2$

	1.260	
1.000	1.000	1.000
	0.740	

All values in these two arrays are expressed in terms of $\frac{P\,a^2}{32}$. In each array, the value at the point being considered is the average of the four adjacent points—adjusted, in the case of Z, by the value $\frac{u\,\Delta^2}{4}$.

The writer considers it a privilege to discuss a paper into which the authors have put so much effort. He hopes that they will be able to continue the work they have begun so ably so that the profession will have the full benefit of their special talents.

Harry D. Conway[30] and George Winter,[31] M. ASCE.—The writers wish to thank Messrs. Guzmán, Luisoni, and Conwell for their discussions. These contributions add materially to the worth of the paper.

The references to theoretical and experimental researches made in Argentina and Sweden are particularly interesting because they indicate the considerable practical importance of the subject. In so far as experimental research is concerned, the difficulty in obtaining a close approximation to uniform loading is considerable. As mentioned in the paper, theory shows that singularities exist in the shearing stresses at points on the boundary where the loading is discontinuous. In practice, the stress gradients are quite likely to be large at these points but they are hardly likely to be infinite. These and other practical considerations must be borne in mind when comparing theory with experiment, and may account for some apparent discrepancies.

The use of the Galerkin variational method by Messrs. Guzmán and Luisoni [20,21] indicates an increase in accuracy with a decrease in labor. However, it is a little surprising to find that, in a later paper,[22,23] they have reverted to the finite difference method. While on this subject, the writers would like to draw attention to a further method presented in the paper by two of the present authors and Mr. Morgan.[8] An extension of this last method to beams of orthotropic material has been presented in a recent paper by one of the writers.[32]

Mr. Conwell's summary, as well as his suggestions for increasing the accuracy of the shear stresses and possibly simplifying the numerical procedures, is valuable. These factors will be carefully considered when making further calculations.

The writers are not familiar with the Swedish[25] and Argentine[26] publications quoted by Messrs. Guzmán and Luisoni in spite of their extensive search of foreign literature. In addition, the second of these[26] was published subsequent to the writing of the present paper. The quotation from the paper by Messrs. Nylander and Holst reinforces the writers' contention that reinforcement should be arranged to counteract cracking, which cannot be achieved by concentrating tension reinforcement at the edge. If the latter is done, cracking converts the structure into an arch-like member whose action is entirely different from that predicted by deep-beam analysis. This, too, seems to be confirmed in the summary by Messrs. Guzmán and Luisoni of the Argentine[26] test results. Even though such arch action subsequent to extensive cracking is stated to require less reinforcement than computed by previous methods, the control over crack formation, possibly even at design loads, seems to be lost

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[32] "Some Problems in Orthotropic Plane Stress," by Harry D. Conway, Paper No. 52-A-4, *Journal of Applied Mechanics*, ASME (publication pending).

in the process—a circumstance which is considerably more serious in deep beams than in shallow beams.

Also, because Messrs. Guzmán and Luisoni state that in these tests "*** failure occurs when the member has lost its resistance to compression in the supports ***" it seems evident that, in view of this extraneous type of failure, full deep-beam (or arch) stresses leading to failure could not have been developed and, therefore, conclusions regarding them must be taken with reservations. The concentration of compression at, and near, the supports may require special measures in the form of reinforcement or even thickening of the section. Only when such local crushing has been eliminated will it be possible, in tests, to determine the effects of various types of reinforcement. Until such more extensive test evidence is available, the indication by Messrs. Guzmán and Luisoni of the importance of compression at the supports should be heeded in design.

In conclusion, attention may be drawn to another paper on this problem, which contains information supplementing in part that presented by the writers.[33]

[33] "The Theory of Girder Walls with Special Reference to Reinforced Concrete Design," by H. L. Uhlmann, *The Structural Engineer*, Vol. XXX, No. 8, London, England, August, 1952, p. 172.